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Metastability for the Exclusion Process with Mean-Field Interaction

Amine Asselah[1] and Giambattista Giacomin[2]

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We consider an exclusion particle system with long-range, mean-field-type interactions at temperature $1/\beta$. The hydrodynamic limit of such a system is given by an integrodifferential equation with one conservation law on the circle $\mathscr{C}$: it is the gradient flux of the Kac free energy functional F_β. For $\beta \leqslant 1$, any constant function with value $m \in [-1, +1]$ is the global minimizer of F_β in the space $\{u : \int_{\mathscr{C}} u(x)\, dx = m\}$. For $\beta > 1$, F_β restricted to $\{u : \int_{\mathscr{C}} u(x)\, dx = m\}$ may have several local minima: in particular, the constant solution may not be the absolute minimizer of F_β. We therefore study the long-time behavior of the particle system when the initial condition is close to a homogeneous stable state, giving results on the time of exit from (suitable) subsets of its domain of attraction. We follow the Freidlin–Wentzell approach: first, we study in detail F_β together with the time asymptotics of the solution of the hydrodynamic equation; then we study the probability of rare events for the particle system, i.e., large deviations from the hydrodynamic limit.

KEY WORDS: Phase segregation models; exclusion processes; mean-field interaction; nonlocal evolution equation; large deviations; Freidlin–Wentzell theory; metastability.

1. INTRODUCTION

Continuum (PDE) models of phase transitions have been proposed since a long time (like for example the Allen–Cahn and the Cahn–Hilliard equations, see, e.g., ref. 15 and references therein) to model a variety of dynamical phenomena involving two or more *phases* or constituents. However, in many of the phase segregation phenomena, like nucleation and metastability, randomness plays a crucial role. In order to model these phenomena

[1] Department Mathematik, ETH-Zentrum, CH-8092 Zürich, Switzerland.
[2] Département de Mathematiques, EPFL, CH-1015 Lausanne, Switzerland.

one can insert some *ad hoc* stochastic terms in the original PDE: this has been done in a variety of ways (see, e.g., ref. 15). However, the choice of the stochastic terms in modelling a real system requires a deep physical insight and it is certainly not easy to choose *one* type of stochastic term that will reproduce the *whole* spectrum of phenomena of the real system we wish to model (see ref. 12 for a review). Also, *random effects* should be the remnance of microscopic effects. As such, they act on a very fine scale and are extremely irregular, so that the physically suggested stochastic modifications of a PDE, typically of white noise type, are often not well posed.

An alternative and, in a sense, more satisfactory way is to look directly at some stochastic particle model, whose evolution law in the large scale limit is a PDE, and for which it makes immediate sense to talk about fluctuations and stochastic effects, on a finer scale.

The progress made in the study of the large scale behavior of stochastic particle systems, (see e.g., ref. 20), has made possible to attack directly from a microscopic standpoint the problems. Very detailed results are already available in the case of systems without a conservation law, see for example refs. 5, 6 and 3. The particle system in this case is a *spin flip* dynamics with long range (mean field type) interactions. Instead, the case with one conservation law is much less developed. We focus here on an exchange process with mean field type interactions.

In 1991, Lebowitz, Orlandi and Presutti studied an exclusion process on $\mathbb{Z}$ with long range interaction at temperature β^{-1}. More precisely, the interaction is 2-body and given by $J_N(x, y) = N^{-1}J((x-y)/N)$, with $x, y \in \mathbb{Z}$. J is a smooth, compactly supported positive function of integral 1. They proved, among other things, that suitable empirical averages on the space scale $N^{1+\delta}$ and time scale $N^{2(1+\delta)}$ converge to the solution of the diffusion equation

$$\partial_t u = \nabla[D_\beta(u)\,\nabla u] \tag{1.1}$$

with $D_\beta(u) = [1-\beta(1-u^2)]/2$ and u_o a smooth initial condition with $D_\beta(u_o(r)) > 0$ for all r (and thus, $D_\beta(u(r,t)) > 0$ for all r and t, see ref. 16). When $\beta > \beta_c \equiv 1$, $D_\beta(u) \leqslant 0$ in an interval $[-m_\beta^*, +m_\beta^*]$ (the *forbidden interval*). The system was derived as a first order expansion (see ref. 16) of a dynamics which is reversible with respect to the local mean field Gibbs measure with interaction J_N.[17, 20] Here we just recall a few basic facts: the *constrained* free energy[17] associated with this measure is

$$f_\beta(u) = \frac{1}{\beta}\left[\left(\frac{1+u}{2}\right)\log\left(\frac{1+u}{2}\right) + \left(\frac{1-u}{2}\right)\log\left(\frac{1-u}{2}\right)\right] - \frac{u^2}{2} \tag{1.2}$$

which has a symmetric double-well structure if $\beta > 1$. The two minima m_β^+ and $-m_\beta^+$ are called phases and $m_\beta^+ > m_\beta^*$. The values $\pm m_\beta^*$ are the inflection points of f_β, so that f_β is convex when restricted to $[-1, -m_\beta^*]$ or $[m_\beta^*, 1]$. There is a standard terminology for this equilibrium model: $[-m_\beta^+, -m_\beta^*) \cup (m_\beta^*, m_\beta^+]$ is called the *metastable region*, and $[-1, -m_\beta^+] \cup [-m_\beta^+, 1]$ is called the *stable region*. The convergence to the solution of (1.1) holds in the stable and metastable regions, but there is no trace in (1.1) of the *transition* between these regions. It is certainly expected that the behavior of the particle system at a very large time will be different in the stable and metastable regions.[17] In the metastable region, it is expected that the system started on a homogeneous profile, eventually rearranges itself into regions in which $u \approx \pm m$, separated by interfaces.

In 1994, Yau[21] studied the hydrodynamic limit for a continuous spin model with local mean field interaction and established the validity of a nondegenerate diffusion equation similar to (1.1), for initial conditions analogous to these in ref. 16. However, he could also prove the validity of such an equation for very large times: up to $t = O(\exp(N^\varepsilon))$, for some $\varepsilon > 0$. Yau conjectured that the result could be improved up to choosing $\varepsilon = 1$ (at a more physical level this had been observed earlier by Lebowitz and Penrose[19]).

The relevance of the integral equation

$$\frac{\partial}{\partial t} u(x, t) = \frac{1}{2} \nabla \left[\nabla u(x, t) - \beta(1 - u^2(x, t)) \int \nabla J(x - y)\, u(y, t)\, dy \right] \tag{1.3}$$

(or its analog for ref. 21) was recognized in both refs. 16 and 21. Equation (1.3) describes the behavior of the system on space scale N and time scale N^2. When $J \geqslant 0$, there are two competing effects: the diffusion and a nonlocal advection term. Note that if J is taken to be a delta-distribution, then Eq. (1.3) becomes (1.1).

In the particle systems in refs. 16 and 21 there are three distinct spatial scales:

(i) *The microscopic scale*, i.e., the lattice scale;

(ii) *The mesoscopic scale*, i.e., the scale of the potential (distances of $O(N)$ lattice sites);

(iii) *The macroscopic scale*, i.e., the scale (or family of scales) in which the range of the potential is very small (for example $O(N^{1+\delta})$ lattice sites, for some $\delta > 0$).

We study the conjecture that we mentioned above[21] on the mesoscopic scale where we average the density over a size of the order of the range of the potential. More precisely, we provide estimates on the exit time from suitable neighborhood of a homogeneous profile m in the metastable region.

Our work parallels Comets',[3] which deals with the metastability problem for the analogous model *without* conservation law. The same type of study can be done about any profile which is a minimum of F_β in the weak topology. We have not characterized the set of minima of F_β and we believe that it is a challenging problem.

In ref. 10 it is observed how the evolution Eq. (1.3), when considered in finite volume, for example on a torus $\mathscr{C}$, is the gradient flux of a free energy,

$$\partial_t u = \nabla \cdot \left[\beta(1-u^2)\, \nabla \frac{\delta F_\beta}{\delta u}(u) \right] \tag{1.4}$$

where

$$F_\beta[u] = \int_{\mathscr{C}} f_\beta(u(x))\, dx + \frac{1}{4} \int_{\mathscr{C} \times \mathscr{C}} J(x-y)[u(x)-u(y)]^2\, dx\, dy \tag{1.5}$$

This clarifies the formal connection between this model and the standard PDE models, notably the Cahn–Hilliard equation, which is just (1.4) with F_β equal to

$$\int_{\mathscr{C}} (f(u(x)) + \zeta\, |\nabla u(x)|^2)\, dx \tag{1.6}$$

where f is a symmetric double well potential (we can take f_β itself, with $\beta > 1$) and ζ is a positive parameter. We mention here common features between the Cahn–Hilliard equation and (1.3), which are physically relevant:

— When studied on the whole line, (1.3) has a unique stationary front joining the two pure phases.[4] This front is easily proved to be stable[1] (compare with ref. 2 where the Cahn–Hilliard equation is treated).

— The Cahn–Hilliard equation and (1.3) have a very close behavior in the so called *sharp interface* limit (see ref. 11 and references therein).

Therefore, we can view the particle model we are discussing as a good candidate for a stochastic modification of the Cahn Hilliard equation.

We remark briefly that, unlike the traditional formulation of the Cahn–Hilliard equation, the evolution equation (1.3), as well as the underlying particle system, includes also antiferromagnetic or partly antiferromagnetic systems (this is the case in which $J(x) \leqslant 0$, for some or all values of x), with their peculiar phenomenology.

2. THE MODEL AND THE MAIN RESULT

We take a system of particles hopping on $V_N = (\mathbb{Z}/N\mathbb{Z})$. A particle configuration is an element $\eta \in X_N = \{-1, +1\}^{V_N}$ and the dynamics is given by the generator which is defined for all $g: X_N \to \mathbb{R}$ as

$$(L_N g)(\eta) = \sum_{i,j} c_{i,j}(\eta)[g(\eta^{i,j}) - g(\eta)] \tag{2.1}$$

where $\eta^{i,j}$ is the configuration η with the i and j entries exchanged and $c_{i,j}(\eta) = 0$ if $|i-j| \neq 1$ and otherwise

$$c_{i,j}(\eta) = \frac{N^2}{2} \exp\left\{-\frac{\beta}{2}[H(\eta^{i,j}) - (H(\eta))]\right\} \tag{2.2}$$

where

$$H_N(\eta) = -\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} J\left(\frac{i-j}{N}\right) \eta_i \eta_j \tag{2.3}$$

where $J \in C^2(\mathscr{C}; \mathbb{R})$ and $J(x) = J(-x)$ for all $x \in \mathscr{C}$. For simplicity, we assume that $J \geqslant 0$, $\int_{\mathscr{C}} J = 1$ and that $J(x)$ is non increasing for $x \in [0, 1/2]$, with $J(1/2) = 0$. The dynamics we just introduced is finite dimensional and therefore can be constructed directly from a finite collection of exponentially distributed variables (see ref. 20, p. 159, or ref. 18), once an initial condition $\eta(0)$ is given. The stochastic process generated by L_N will be denoted by $\{\eta(t)\}_{t \in \mathbb{R}}$ and $\eta(\cdot) \in D([0, \infty); X_N)$, the Skorokhod space of X_N-valued functions. It can be easily verified that L_N is self-adjoint in $L^2(X_N; d\nu_N)$, where ν_N is any element in the family of probability measures indexed by the parameter $\lambda \in \mathbb{R}$, defined as

$$\exp\left\{-\beta H_N(\eta) - \beta\lambda \sum_i \eta_i\right\} \Big/ Z_N(\beta, \lambda) \tag{2.4}$$

where $Z_N(\beta, \lambda)$ is the normalizing factor. This property is usually referred to as *reversibility* with respect to ν_N and, in general, it plays an important role.[20] In particular, it implies that, for any $\lambda \in \mathbb{R}$, ν_N is an invariant measure for the process $\eta(\cdot)$.

In visual terms, the particles hop on the lattice at Poisson times, with a tendency to clump together (if $J \geqslant 0$) and, in one unit of time, there will be $O(N^2)$ jumps, due to the factor N^2 in front of the rates (2.2).

We are going to observe this system on the spatial scale N, i.e., we will look at the empirical density

$$\mu_N = \sum_{i=1}^{N} \eta_i \mathbf{1}_{[(i-1)/N,\, i/N]} \tag{2.5}$$

and μ_N is an element of M_*, the space of positive measures with density f with respect to the Lebesgue measure on $\mathscr{C}$ such that $f \leqslant 1$. Unless otherwise stated, M_* is equipped with the (metrizable) topology of weak convergence induced by $C^0(\mathscr{C})$ (we denote the duality by $(\cdot, \cdot)$). Therefore $\mu_N(\eta(\cdot)) \in D(\mathbb{R}^+; M_*)$. We will use the notation $u \in M_*$, with the meaning that $u(x)\, dx \in M_*$. We will denote by M_m, $m \in [0, 1]$ the convex set $\{u \in M_* : \int u(x)\, dx = m\}$.

The *hydrodynamic limit* (law of large numbers) for this system (see ref. 10 for a proof) in the case of deterministic initial conditions says that if $\{\eta_N(0)\}_N$ is a sequence such that $\mu_N(\eta_N(0)) \in M_*$ converges to a limit which we represent via its density $u_o \in M_*$, then $\{\mu_N(t)\}_N$ converges to $u(\cdot, t)$ which is the unique weak solution of (1.3).

In this paper we prove a Large Deviation Principle from the hydrodynamic limit for general J and apply it to understand the long time dynamics in the ferromagnetic case ($J \geqslant 0$). This is particularly interesting when $u_0(x) = m$ for all $x \in \mathscr{C}$ and m is in the metastable region which we define as the set of constant profiles m which are strict local minimizers in the weak topology of F_β among all functions $u \in M_m$, but not absolute minima in the same class. In the stable region, F_β constrained to M_m is strictly convex and has therefore only one minimizer.

The link between the dynamical large deviation functional (which we will call $I_T(u)$ for $T > 0$ and $u \in D([0, T]; M_*)$) is provided by the solution of the *quasipotential problem.*[8] We show that for suitable $\xi \in M_m$ that

$$\inf\{I_T(u) : u(0, \cdot) \equiv m,\, u(T, \cdot) = \xi(\cdot),\, T > 0\} = F_\beta[\xi] - F_\beta[m] \tag{2.6}$$

We are then addressing the problem of the *exit from a domain*:[8] if τ_N is the first exit time from the basin of attraction of m, then there exists $c > 0$ such that for N sufficiently large

$$E(\tau_N) \geqslant \exp\{cN\} \tag{2.7}$$

thus answering the problem raised in ref. 21 in the restricted case of volumes of mesoscopic size. This strongly suggests that a statement like (2.7) should be true also if the volumes are of macroscopic size, for example $|V_N| = O(N^{1+\delta})$ for any $\delta > 0$, but to obtain such a result requires stronger probability estimates on the system.

We will prove a result much stronger than (2.7), but only for a family of domains in M_*. We now explain in more details this result. The problem is that the energy is not continuous in the weak topology but only lower semi-continuous. Thus, we need neighborhoods D of m such that in any small ball B around one minimizer of F on ∂D, say u^*, there are profiles $u \in B \cap D^c$ such that $F(u)$ is close to $F(u^*)$. We cannot show that this property holds for any D but rather build a large collection of such *good* D. For m and $\gamma > 0$ such that any $u(\cdot) \equiv m_o \in [m-\gamma, m+\gamma]$ is metastable, we show that there is a dense set $\mathscr{D}$ of positive reals and a family, $\{\Gamma_{T,\gamma}$, for $T \in \mathscr{D}\}$, of weak closed neighborhoods of $E = \{u \in M_* : u(\cdot) \equiv m \in [m-\gamma, m+\gamma]\}$ attracted to E with the property that if $T' < T$, then $\Gamma_{T',\gamma} \subset \text{int}_w \Gamma_{T,\gamma}$ (where int_w is the weak interior). Now we can define for a path $u \in D([0, \infty), M_1)$, the stopping time

$$\tau_\gamma(u) = \inf\{t : u(t) \notin \Gamma_{T,\gamma}\}$$

Finally, if $\partial\Gamma_{T,\gamma}$ is the boundary of $\Gamma_{T,\gamma}$ and $\partial_m \Gamma_{T,\gamma} \equiv \partial\Gamma_{T,\gamma} \cap \{u : \int u = m\}$, our main result, proved in Section 7, reads

Theorem 2.1. For $\varepsilon > 0$, there is γ_o such that if $\gamma < \gamma_o$, $T_\varepsilon < T - \varepsilon$ and N large enough, then for any $\eta_N \in \{\mu_N(\eta_N) \in \Gamma_{T_\varepsilon,\gamma}\}$

$$\left| \frac{1}{N} \log E_N^{\eta_N}[\tau_\gamma] - \inf_{\partial_m \Gamma_{T,\gamma}} (F[\xi] - F[m]) \right| \leqslant \varepsilon \tag{2.8}$$

Notations. We denote the constant function $u(x) \equiv m$, for $x \in \mathscr{C}$ simply by m. Also, besides its standard meaning, we will often use the notation $[a, b]$ $(-1 \leqslant a < b \leqslant 1)$ to indicate the set $\{u \in M_* : u(\cdot) \equiv m, a \leqslant m \leqslant b\}$. In the sequel, we drop the index β (e.g., $m^+ = m_\beta^+$, $m^* = m_\beta^*$, and so on) and we write the energy as

$$F[u] = \frac{1}{\beta} \int_{\mathscr{C}} \phi(u(x))\, dx - \frac{1}{2} \int_{\mathscr{C}} \int_{\mathscr{C}} J(x-y)\, u(x)\, u(y)\, dx\, dy \tag{2.9}$$

with

$$\phi(u) = \left(\frac{1+u}{2}\right) \log\left(\frac{1+u}{2}\right) + \left(\frac{1-u}{2}\right) \log\left(\frac{1-u}{2}\right) \tag{2.10}$$

We will denote by $\{\varphi_\delta\}_{\delta>0}$ an approximate identity on $\mathscr{C}$. By this we mean that for each $\delta>0$, φ_δ is even, C^∞, supported on $(-\delta, \delta) \subset \mathscr{C}$ and $\lim_{\delta \to 0} \int \varphi_\delta f \, dx = f(0)$ for every $f \in C^0(\mathscr{C})$.

In order to keep the notation light, we will write the evolution equations always in strong form, even when they have to be interpreted in the weak sense: the weak form is obtained by integrating against a function $G \in C^{1,2}([0, T]; \mathscr{C})$ and performing the formal integration by parts.

The paper is carried out for simplicity in the case of $d=1$. In no argument this is essential: the probability arguments require no modifications, while the PDE arguments require sometimes better Sobolev estimates, which can be done straightforwardly. In case of $d>1$ the 2-body potential would be $J(\cdot/N)/N^d$ and the escape time in Theorem 2.1 is of order $\exp(const.N^d)$.

Plan of the Paper. In Section 3 we establish that all possible limits in time of the solution of (1.3) on the circle $\mathscr{C}$, are extremal points of the energy. This section is not properly essential to the derivation of the main result, except Lemma 3.2, Lemma 3.3 and Corollary 3.7.

In Section 4, we define and characterize the metastable region and show that there, homogeneous densities are asymptotically stable for (1.3) in the weak topology. Then, we focus on a homogeneous profile, say of density m, and consider a small neighborhood D of m, such that m is the only extremal point of F in D.

In Section 5, we prove Large Deviation estimates: that is, we estimate the probability of being in small tubes (in the weak topology) about some trajectories linking m to the boundary of D; the estimates we obtain are uniform in the initial configuration (in D). We have built on the work of Kipnis, Olla and Varadhan in ref. 14. We mention that our rate function is not convex, and that lower semi-continuity is not obvious as in ref. 14.

In Section 6, we build neighborhoods of metastable states with desirable properties. Also, we establish several properties of the rate function.

Finally, in Section 7, we derive our main result on the exit time.

3. THE HYDRODYNAMIC EQUATION AND ITS ASYMPTOTIC BEHAVIOR

In this section we study some properties of the solutions of

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} u - \beta(1-u^2) \frac{\partial}{\partial x} (J * u) \right] \tag{3.1}$$

for u_o of average $m \in [-1, 1]$. We denote by $u(u_0, t)$ the solution of (3.1) at time t with initial condition u_o. Existence and uniqueness result for (3.1) on the whole line were obtained in ref. 4 (see ref. 10 for the case of the circle). Henceforth, we define by

$$M_m = \left\{ u \text{ measurable} : u \in [-1, 1], \int u = m \right\}$$

The main result of this section is the following:

Theorem 3.1. For any $u_0 \in M_m$, $\{u(u_0, t)\}_{t \geqslant 0}$ is relatively compact in $L^\infty(\mathscr{C})$. For any limit point $u^* \in M_m$ there exists $\lambda \in \mathbb{R}$ such that

$$u^* = \tanh(\beta J * u^* + \lambda) \tag{3.2}$$

and $u^* \in C^\infty(\mathscr{C})$.

Strictly speaking, this result will not be used in the text. However it requires little more effort than the minimal results on the time asymptotics of (3.1) needed in the later sections and it gives a more complete picture.

The key fact that we will repeatedly use is that the functional in (2.9) is a Lyapunov functional for (3.1). In fact, we will show that if $u_o \in (-1, 1)$, and $t > 0$,

$$\frac{dF[u(u_o, t)]}{dt} = -\int_{\mathscr{C}} (1 - u^2(u_o, t)) \left[\frac{\partial}{\partial x} \frac{\delta F}{\delta u} [u(u_o, t)] \right]^2 dx \tag{3.3}$$

One difficulty arises from the behavior of the equation when u takes on the values ± 1: in particular one can verify that F is not Fréchet differentiable at any (smooth) u such that $|u(x)| = 1$ for some $x \in \mathscr{C}$.

A crucial result for us is the following:

Lemma 3.2. For $T > 0$, the map $u_o \mapsto u(u_o, T)$ is continuous from M_* with the weak topology to M_* with the L^∞ topology.

Proof. Let u and v^n be solutions of (3.1) with initial condition u_o and v_o^n respectively, and assume $\{v_o^n\}$ converges weakly to u_o. If $\Psi(t, x)$ is the heat kernel in $\mathscr{C}$, u can be represented as

$$u(x, t) = \Psi(t) * u_o(x) + \beta \int_0^t \Psi_x(t - s) * [(J_x * u)(1 - u^2)(s)](x)\, ds \tag{3.4}$$

and similarly for v^n. If $w^n = v^n - u$, then

$$w^n(x, t) = \Psi(t) * w_o^n(x) + \beta \int_0^t \Psi_x(t-s) * [J_x * w^n(1-u^2)(s) + J_x * v^n(u^2 - (v^n)^2)](x) \tag{3.5}$$

Now, since w^n is uniformly bounded, there is a constant C_1 such that

$$|w^n(t)|_\infty \leqslant |\Psi(t) * w_o^n|_\infty + C_1 \int_0^t |\Psi_x(t-s)|_1 \, |w^n(s)|_\infty \, ds$$

Now, for an arbitrary small t_1, the classical bound $|\Psi_x(t)|_1 \leqslant C_2/\sqrt{t}$ gives us

$$\begin{aligned} |w^n(t_1)|_\infty &\leqslant |\Psi(t_1) * w_o^n|_\infty + 2C_1 C_2 \sup_{s \leqslant t_1} |w^n(s)|_\infty \sqrt{t_1} \\ &\leqslant |\Psi(t_1) * w_o^n|_\infty + C\sqrt{t_1} \end{aligned}$$

We can rewrite (3.5) between time t_1 and t, and if we recall that $|\Psi(t-t_1)|_1 = 1$ then,

$$|w^n(\cdot, t_1 + t)|_\infty \leqslant |w^n(\cdot, t_1)|_\infty + C_1 \int_0^t \frac{1}{\sqrt{t-s}} |w^n(\cdot, t_1 + s)|_\infty \, ds \tag{3.6}$$

Now, for $\alpha > 0$, and any function $g(x, t)$ define

$$\|g\|_{\alpha, t} = \sup_{s \leqslant t} e^{-\alpha s} |g(\cdot, s)|_\infty$$

Clearly, $t \mapsto \|g\|_{\alpha, t}$ is increasing. We can multiply both sides of (3.6) by $\exp(-\alpha t)$ to obtain

$$\begin{aligned} |w^n(\cdot, t + t_1)|_\infty e^{-\alpha t} &\leqslant |w^n(\cdot, t_1)|_\infty e^{-\alpha t} + C_1 \int_0^t \|w^n(\cdot, \cdot + t_1)\|_{\alpha, s} \frac{e^{-\alpha(t-s)}}{\sqrt{t-s}} \, ds \\ &\leqslant |w^n(\cdot, t_1)|_\infty e^{-\alpha t} + C_1 \|w^n(\cdot, \cdot + t_1)\|_{\alpha, t} \int_0^\infty \frac{e^{-\alpha s}}{\sqrt{s}} \, ds \end{aligned} \tag{3.7}$$

We can choose α_0 such that $C_1 \int \exp(-\alpha_o s)/\sqrt{s} = 1/2$ and take a supremum over time up to t on both sides of (3.7),

$$\|w^n(\cdot, \cdot + t_1)\|_{\alpha, t} \left(1 - C_1 \int_0^\infty \frac{e^{-\alpha_o s}}{\sqrt{s}} ds\right) \leqslant |w^n(\cdot, t_1)|_\infty$$

In other words,

$$|w^n(\cdot, T)|_\infty \leqslant 2(|\Psi(t_1) * w_o^n|_\infty + C\sqrt{t_1})\, e^{\alpha_o T} \tag{3.8}$$

Now, for any $\varepsilon > 0$, let t_1 be small enough so that $2C\sqrt{t_1}\exp(\alpha_o T) < \varepsilon$, then choose n large enough so that $2\,|\Psi(t_1) * w_o^n|_\infty \exp(\alpha_o T) < \varepsilon$. Then (3.8) implies that $|w^n(\cdot, T)|_\infty \leqslant 2\varepsilon$. ∎

We will need some uniform control on the behavior of the derivatives of $u(u_o, t)$. We collect these results in the next two Lemmas.

Lemma 3.3. For any $t_o > 0$ and n, there is $C(n, t_o) > 0$ such that for any $u_o \in M_m$ and $t \geqslant t_o$

$$|D^{(n)} u(u_o, t)|_2^2 \leqslant C(n, t_o) \tag{3.9}$$

Remark 3.4. It is straightforward to perform parabolic estimates (like for example in ref. 13) to show that if $u_o \in C^\infty(\mathscr{C})$, then $u(u_o, t) \in C^\infty(\mathscr{C})$ for any $t \geqslant 0$.

Proof. Since the case $|m| = 1$ is trivial, we will assume $|m| < 1$. We first assume that $u_o \in C^\infty$. Remark 3.4 allows us then to use (3.1) and integrate by parts. Also, we start with the cases $n = 1$ and 2 and explain how the general case follows.

Suppose that there is $C > 0$ such that for any $t > 0$

$$\frac{d}{dt}|u(t) - m|_2^2 + \frac{1}{2}|u_x(t)|_2^2 \leqslant C \tag{3.10}$$

and,

$$\frac{d}{dt}|u_x(t)|_2^2 \leqslant C\,|u_x(t)|_2^2 \tag{3.11}$$

Suppose also that there is $S > t_o$ such that

$$|u_x(S)|_2^2 \geqslant 2(C + 5/t_o)\, e^{Ct_o}$$

For any $t \in [S-t_o, S)$ we can integrate (3.11) between $S-t$ and S to obtain

$$|u_x(S-t)|_2^2 \geqslant |u_x(S)|_2^2 \, e^{-Ct_o} \geqslant 2(C+5/t_o) \tag{3.12}$$

and thus by (3.10),

$$\frac{d}{dt} |u(t)-m|_2^2 \leqslant -5/t_o$$

which after integration yields

$$|u(S)-m|_2^2 \leqslant |u(S-t_o)-m|_2^2 - 5t_o/t_o \leqslant -1 \tag{3.13}$$

which is absurd. Thus, we can take $C(1, t_o) = 2(C+5/t_o)\, e^{Ct_o}$.

Now, we prove (3.10) and (3.11). We multiply (3.1) by $u-m$ and integrate by parts

$$\frac{d}{dt} |u(t)-m|_2^2 + |u_x(t)|_2^2 = \int (J_x * u)\, u_x(1-u^2)\, dx$$

By Cauchy–Schwarz inequality

$$\frac{d}{dt} |u(t)-m|_2^2 + \frac{1}{2} |u_x(t)|_2^2 \leqslant \frac{1}{2} \int (J_x * u)^2 \, dx \leqslant \frac{1}{2} |J_x|_1^2$$

Now, we multiply (3.1) by u_{xx} and integrate by parts

$$\frac{d}{dt} |u_x(t)|_2^2 + |u_{xx}(t)|_2^2$$
$$\leqslant \int |J_x * u_x| \, |u_{xx}| \, dx + 2 \int |(J_x * u)\, u_x u| \, |u_{xx}| \, dx \tag{3.14}$$

then by Cauchy–Schwarz inequality

$$\frac{d}{dt} |u_x(t)|_2^2 \leqslant \frac{1}{2} \int |J_x * u_x(1-u^2)|^2 \, dx + 2 \int |(J_x * u)\, u_x u|^2 \, dx \leqslant 3\, |J_x|_1^2 \, |u_x|_2^2$$

and we can set $C = 3\, |J_x|_1^2$ and obtain (3.10) and (3.11). Note that we only used the structure of (3.1) and the boundedness of $|u(t)|_2$. We now derive the same equations for u_{xx} using that $|u_x(t)|_2^2 \leqslant C(1, t_o)$ when $t \geqslant t_o > 0$.

We differentiate (3.1), multiply by D^3u and integrate by parts

$$\frac{d}{dt}|u_{xx}(t)|_2^2 + |D^3u(t)|_2^2$$

$$= \int [J_x * u_x(1-u^2) - 2(J_x * u)\, u_x u]_x\, D^3u$$

$$\leqslant \int |J_x * D^3u(1-u^2) - 4J_x * u_x u_x u + (J_x * u)\, u_{xx} - 2(J_x * u)\, u_x^2|\, |D^3u| \tag{3.15}$$

It is now easy to obtain from (3.14) and (3.15) that for $t \geqslant t_o$ and some $K > 0$

$$\frac{d}{dt}|u_x(t)|_2^2 + \frac{1}{2}|u_{xx}(t)|_2^2 \leqslant K \qquad \text{and} \qquad \frac{d}{dt}|u_{xx}(t)|_2^2 \leqslant K\, |u_{xx}(t)|_2^2 \tag{3.16}$$

Similarly, for $t \geqslant t_o > 0$, there is $C(2, t_o)$ such that $|u_{xx}(t)|_2^2 \leqslant C(2, t_o)$. The inequality for any n is obtained by taking further derivatives and integration by parts, and by repeating the same procedure.

To treat now the case of a general initial condition, note that $u \mapsto |D^k u|_2$ is a weakly l.s.c. functional. Indeed, it can be written as a supremum over continuous functionals

$$\int (D^n u)^2 = \sup_{\varphi} \left[(-1)^n \int u D^n \varphi - \tfrac{1}{2} \int \varphi^2 \right]$$

Therefore, since $u_o^n = \alpha_n * u_o$ is C^∞ and converges weakly to u_o, Lemma 3.2 tells us that for $t > 0$, $u(u_o^n, t)$ converges to $u(u_o, t)$ in L^∞ (and therefore weakly). By lower semicontinuity

$$\liminf_{n \to \infty} \int (D^k u(u_o^n, t))^2 \geqslant \int (D^k u(u_o, t))^2$$

Hence (3.9) holds for any initial condition. ∎

Lemma 3.5. For any $t_o > 0$, and $u_o \in M_m$ there is $D_1(n, t_o) > 0$ and $D_2(n, t_o) > 0$ such that for $t \geqslant t_o$

$$\left| \frac{\partial}{\partial t} D^n u(t) \right|_2^2 \leqslant D_1(n, t_o) \tag{3.17}$$

and for $t \geqslant s \geqslant t_o$

$$|D^n u(t) - D^n u(s)|_\infty^2 \leqslant D_2(n, t_o)(t-s) \tag{3.18}$$

Proof. We can assume that $|m| < 1$. From Lemma 3.3, $u(u_o, t) \in C^\infty$ for $t \geqslant t_o$. By taking n derivatives of (3.1) and the L^2 norm on both sides and (3.9) we obtain (3.17). By Cauchy–Schwarz inequality

$$|D^n u(t) - D^n u(s)|_2^2 \leqslant (t-s) \int_s^t |D^n u_t(\tau))|_2^2 \, d\tau \leqslant D_1(n, t_o)(t-s)^2 \tag{3.19}$$

By using Agmon inequality, there is $C_2 > 0$ such that for $t \geqslant s \geqslant t_o$

$$|D^n u(t) - D^n u(s)|_\infty^2 \leqslant C_2 \, |D^n u(t) - D^n u(s)|_2 \, |D^n u_x(t) - D^n u_x(s)|_2$$

and (3.18) is obtained when we use (3.19). ∎

Lemma 3.6. If $u_o \in (-1, 1)$ is $C^{2,1}(\mathscr{C})$, then $u(u_o, t) \in (-1, 1)$ for all $t \geqslant 0$.

Proof. It has been established in ref. 4 that $u(u_o, t) \in [-1, 1]$. Also, when u_o is $C^{2,1}(\mathscr{C})$, we can interpret (3.1) in a strong sense. We define $v \equiv 1 - u^2$ and rewrite (3.1) as an equation for v. We multiply both sides of (3.1) by $-2u$ to obtain

$$-2u \frac{\partial u}{\partial t} = -u \frac{\partial^2 u}{\partial x^2} + \beta u(1-u^2) \frac{\partial^2 J * u}{\partial x^2} - 2u^2 \frac{\partial u}{\partial x} \frac{\partial J * u}{\partial x}$$

Now, we use that

$$\frac{\partial v}{\partial t} = -2u \frac{\partial u}{\partial t}, \qquad \frac{\partial v}{\partial x} = -2u \frac{\partial u}{\partial x}, \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} = -2u \frac{\partial^2 u}{\partial x^2} - 2\left(\frac{\partial u}{\partial x}\right)^2$$

to obtain

$$\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \beta v u \frac{\partial^2 J * u}{\partial x^2} + \frac{\partial v}{\partial x}\left(u \frac{\partial J * u}{\partial x}\right) \tag{3.20}$$

Now, for each t, we define $\zeta(t) = \min_{x \in \mathscr{C}} v(t, x)$. We fix a T and take $t \leqslant T$. We assume that the minimum at time t of v is realized at x. Let n be a large integer to be taken to infinity later, and define $\Delta = T/n$. We will often use that v is $C^{2,1}(\mathscr{C})$ on the compact $[0, t] \times \mathscr{C}$. Now, by Taylor expansion, there is C which only depends on T, such that

$$v(t, x) - v(t - \Delta, x) \geqslant \frac{\partial v}{\partial t}(t, x)\, \Delta - C\Delta^2$$

Thus, as $\zeta(t - \Delta) \leqslant v(t - \Delta, x)$ by definition, we have

$$\zeta(t) - \zeta(t - \Delta) \geqslant \frac{\partial v}{\partial t}(t, x)\, \Delta - C\Delta^2$$

Now, from (3.20), there is γ finite independent of t such that

$$2\frac{\partial v}{\partial t}(t, x) \geqslant v(t, x)\, \beta u \frac{\partial^2 J * u}{\partial x^2} \geqslant -2\gamma v(t, x)$$

Thus,

$$\zeta(t) - \zeta(t - \Delta) \geqslant -\gamma\zeta(t)\, \Delta - C\Delta^2$$

We multiply by $\exp(\gamma(t - \Delta))$ both terms of the equation to obtain

$$[e^{\gamma t}\zeta(t)]_{t-\Delta}^{t} \geqslant -\Delta\left(\gamma - \frac{1 - e^{-\gamma\Delta}}{\Delta}\right) e^{\gamma t}\zeta(t) - C\Delta^2 \tag{3.21}$$

Now, we use that ζ is bounded by 1, that $e^{\gamma t} \leqslant e^{\gamma T}$, and that there $C_1 > 0$ such that

$$\left|\gamma - \frac{1 - e^{-\gamma\Delta}}{\Delta}\right| \leqslant C_1\, \Delta$$

We sum (3.21) over all t of the form kT/n for $k \in \{1, ..., n\}$ to obtain

$$e^{\gamma T}\zeta(T) \geqslant \zeta(0) - \frac{T}{n}(C + C_1 e^{\gamma T})$$

The result follows now as $n \to \infty$. ∎

Proof of Theorem 3.1.

Step 1. We have seen that for any $t_o > 0$ and $t \geqslant t_o$, $|u(t)|_2 \leqslant 1$ and $|u_x(t)|_2 \leqslant C(1, t_o)$. This implies that the family $\{u(t), t > t_o\}$ is equicontinuous and uniformly bounded. Ascoli–Arzelà's theorem implies that for any sequence of times going to infinity there is a subsequence $\{t_n\}$ and u^* continuous such that

$$\lim_{n \to \infty} |u(t_n) - u^*|_\infty = 0$$

Now, by symmetry we can assume $m > 0$. If $m < 1$, there is $\varepsilon > 0$ such that for $r \geqslant 0$,

$$r \leqslant 2 + \frac{1}{2\beta} \log\left(\frac{1+m}{1-m}\right) \Rightarrow \tanh(\beta r) \in [-1 + 2\varepsilon, 1 - 2\varepsilon] \tag{3.22}$$

We choose any $\varepsilon > 0$ such that (3.22) holds and define

$$A_n = \{x : 1 - u(t_n, x)^2 > \varepsilon/2\}, \qquad \text{and} \qquad A = \{x : 1 - u^*(x)^2 > \varepsilon\}$$

Note that by Corollary 3.5, for δ small enough and n large, if $|t - t_n| < \delta$, then

$$A \subset A_n \subset \left\{x : 1 - u(t, x)^2 > \frac{\varepsilon}{4}\right\}$$

Step 2. We know by Lemma 3.3 that $u(t_o)$ is $C^{2,1}$. Let the sequence $\{u_o^n \in (-1, 1), n \in \mathbb{N}\}$ be defined by $u_o^n(x) = (1 - 1/n)\, u(t_o, x)$. It is easy to see from Lemma 3.2 that for any $t \geqslant 0$, $u(u_o^n, t)$ and its first derivative converge pointwise to $u(t + t_o)$. The reason to choose such $u_o^n \in (-1, 1)$ is that the kernel of the Fréchet derivative of F is well defined only for $u \in (-1, 1)$

$$\frac{\delta F}{\delta u}[u] = -J * u + \frac{1}{2\beta} \log\left(\frac{1+u}{1-u}\right) \tag{3.23}$$

If we multipy (3.1) by $\delta F/\delta u$ and integrate by parts, we obtain for $t \geqslant 0$,

$$\frac{dF[u(u_o^n, t)]}{dt} = -\beta \int_{\mathscr{C}} (1 - u^2(u_o^n, t)) \left[\frac{\partial}{\partial x} \frac{\delta F}{\delta u}[u(u_o^n, t)]\right]^2 dx$$

If we integrate the previous equation and recall that F is bounded,

$$F[u_o^n] - \inf F \geqslant \int_0^\infty \int (1 - u^2(u_o^n, t)) \left[\frac{\partial}{\partial x} \frac{\delta F}{\delta u}[u(u_o^n, t)]\right]^2 dx\, dt \tag{3.24}$$

Also, if we restrict the space integration over A, and for n_o large enough

$$F[u_o^n]-\inf F \geqslant \sum_{k=n_o}^{\infty} \int_{t_k-\delta}^{t_k+\delta} \int_A (1-u^2(u_o^n, t-t_o)) \left[\frac{\partial}{\partial x} \frac{\delta F}{\delta u}[u(u_o^n, t-t_o)] \right]^2 dx\, dt$$

Now, by the continuity of F in the sup norm and Fatou's Lemma, we obtain

$$\infty > \frac{\varepsilon}{4} \sum_{n=n_o}^{\infty} \int_{t_n-\delta}^{t_n+\delta} \int_A \left[\frac{\partial}{\partial x} \frac{\delta F}{\delta u}[u(u_o, t)] \right]^2 dx\, dt$$

By continuity of u^*, there is x_o such that $u^*(x_o)=m$. If we think of the circle $\mathscr{C}$ as $[-1/2+x_o, 1/2+x_o)$, then we can define

$$a=\inf\{x\in[-1/2+x_o, x_o) : 1-u^*(x)^2>\varepsilon\}$$

and

$$b=\sup\{x\in(x_o, 1/2+x_o] : 1-u^*(x)^2>\varepsilon\}$$

Then $A\supset[a,b]$ and by Poincaré inequality in $[a,b]$

$$\infty > \sum_{n=n_o}^{\infty} \int_{t_n-\delta}^{t_n+\delta} \int_a^b \left(\frac{\delta F}{\delta u}[u(u_o, t)] - \frac{1}{b-a} \int_a^b \frac{\delta F}{\delta u}[u(u_o, t)] \right)^2 dx\, dt$$

Because of the uniform Holder continuity in time in sup norm (Corollary 3.5)

$$\lim_{n\to\infty} \int_a^b \left(\frac{\delta F}{\delta u}[u(u_o, t_n)] - \frac{1}{b-a} \int_a^b \frac{\delta F}{\delta u}[u(u_o, t_n)] \right)^2 = 0$$

Hence, a.s. in $[a,b]$

$$-J*u^* + \frac{1}{2\beta}\log\left(\frac{1+u^*}{1-u^*}\right) = \frac{1}{b-a}\int_a^b \frac{\delta F}{\delta u}(u^*)$$

If we evaluate this expression at x_o and recall that u^* is continuous

$$\frac{1}{2\beta}\log\left(\frac{1+u^*}{1-u^*}\right)(x) = J*u^*(x) - J*u^*(x_o) + \frac{1}{2\beta}\log\left(\frac{1+m}{1-m}\right), \qquad x\in[a,b] \tag{3.25}$$

Because the right hand side is bounded, u^* remains away from -1 and 1, and by (3.22) $\{u^* > \varepsilon\} = [0, 1]$. Also, (3.25) shows that $u^* \in C^\infty$. ∎

We prove now a corollary which will be used in Section 6.

Corollary 3.7. Let $m \in (-1, 1)$, $u_o \in M_m$ and $t > 0$. Assume that

$$\frac{dF[u(u_o, t)]}{dt} = 0 \tag{3.26}$$

then $u(u_o, t)$ is an extremum of F.

Proof. For $t > 0$, $u(u_o, t) \in C^\infty$ and $\int u(u_o, t) = m$. We can define x_o, by $u(t, x_o) = m$ and

$$a = \inf\{x \leqslant x_o : 1 - u(t, x)^2 > \varepsilon\},$$

and

$$b = \sup\{x \geqslant x_o : 1 - u(t, x)^2 > \varepsilon\}$$

Now, as in the previous proof, our hypothesis (3.26) implies that for $x \in [a, b]$

$$-J * u(t, x) + \frac{1}{2\beta} \log\left(\frac{1 + u(t, x)}{1 - u(t, x)}\right) = \lambda, \qquad \text{with} \quad |\lambda| \leqslant 1 + \frac{1}{2\beta} \log\left(\frac{1 + m}{1 - m}\right)$$

Thus, for any $x \in [a, b]$, $\tanh(\beta(\lambda + J * u(t, x))) = u(t, x)$ and by (3.22) $u(t, x) \in [1 - 2\varepsilon, 1 + 2\varepsilon]$. Thus, $[a, b] = [0, 1]$ and the result follows. ∎

4. THE METASTABLE REGION

We have seen in Section 3 that our dynamics is associated with a Lyapunov functional F. In this section, we study some extrema of F given in (2.9) in M_m. When $u \equiv m$, then $m \mapsto F[m] = \phi(m)/\beta - m^2/2$ is smooth and even. If $\beta > 1$, $F[m]$ has a double well shape with minima at $\pm m^+$ where $F'(m^+) = 0$ and inflexion points at $\pm m^*$ where $F''(m^*) = 0$. From a physical point of view, if the convex hull of F is the free energy of a macroscopic system, then a homogeneous density $u \equiv m$ with $m \in (-m^+, m^+)$ is not an equilibrium state. Accordingly, we define the metastable region to be the set of $m \in (-m^+, m^+)$ such that $u \equiv m$ is a local minimum of F in M_m. The concept of local minimum requires that we specify the topology. Indeed, the result that we need would be trivial if we were to use the supremum norm topology. However, the Large Deviation estimates require us to work with the weak topology.

The main result of this section is the characterization of the metastable region. We actually provide two proofs of this result: a direct proof in Theorem 4.1, which gives as a useful corollary that in a small L^2 neighborhood of m, $F[u]-F[m]$ is equivalent to $|u-m|_2^2$. The second proof (Lemma 4.3) relies on an integration by parts of (3.1) and Lemma 3.2.

We can define $\tilde{m}\in(0, m^+)$ as

$$\frac{1}{\beta}\phi''(\tilde{m})=\sup_{k\geqslant 1}\int_0^1 \cos(2\pi kx)\, J(x)\, dx \tag{4.1}$$

It is then easy to see that for any $m\geqslant\tilde{m}$ and $\int f=0$, (4.1) implies that

$$-\iint J(x-y)\, f(x)\, f(y)\, dx\, dy+\frac{1}{\beta}\phi''(m)\int f(x)^2\, dx\geqslant 0 \tag{4.2}$$

Theorem 4.1. For $\beta>1$, any homogeneous density $m\in[-m^+, -\tilde{m})\cup(\tilde{m}, m^+]$ is a strict local minimizer of F in the weals topology among elements of M_m.

Proof.

Step 1. We show that if m is a strict local minimizer in L^2, then it is a strict local minimizer weakly. Assume that $F(m)$ is not a strict local minimum weakly. In other words, there exists a sequence $\{f_n\}$ with

$$-1\leqslant f_n+m\leqslant 1, \qquad \int f_n=0,\ \lim_{n\to\infty} f_n=0 \text{weakly, but } F(m+f_n)\leqslant F(m)$$

In conjunction with l.s.c. of F,[3] this implies that

$$\lim_{n\to\infty} F(m+f_n)=F(m) \tag{4.3}$$

Now, $u\mapsto(J*u, u)$ is weakly continuous so (4.3) implies that

$$\lim_{n\to\infty}\int \phi(m+f_n)-\phi(m)=0 \tag{4.4}$$

We rewrite

$$\int\phi(m+f_n)-\phi(m)=\frac{1+m}{2}\int\psi^+\log(\psi^+)+\frac{1-m}{2}\int\psi^-\log(\psi^-)$$

with

$$\psi^+ = 1 + \frac{f_n}{1+m}, \qquad \text{and} \qquad \psi^- = 1 - \frac{f_n}{1-m}$$

Also, one can check by Taylor expansion that for $-1 \leqslant x \leqslant M$,

$$x + x^2 \geqslant (x+1)\log(x+1) \geqslant x + \frac{x^2}{2(1+M)} \tag{4.5}$$

Thus,

$$\frac{1}{1-m^2}\int f_n^2 \geqslant \int \phi(m+f_n) - \phi(m) \geqslant \frac{1}{2(1+M)}\frac{1}{1-m^2}\int f_n^2 \tag{4.6}$$

with $M = (1+m)/(1-m)$. Now, (4.4) and (4.6) imply that $\{f_n\}$ goes to 0 in L^2 and therefore m is not a strict local minimum in L^2.

Step 2. We assume now that f is small in L^2, and we will rewrite F separating the contributions of the small and large values of f.

Because $m > \tilde{m}$, we can choose $\delta > 0$ such that $\delta \leqslant 2(1-m)$ and

$$(1-\delta)\left[\phi''(m) - \frac{\delta}{3}\left(\frac{1}{(1+m)^2} + \frac{1}{(1-m)^2}\right)\right] \geqslant \phi''(\tilde{m}) \tag{4.7}$$

If we define $A = \{x : |f| > \delta\}$, then $|A|$ goes to 0 with $|f|_2$ and

$$|A| \leqslant \frac{1}{\delta^2}\int f^2 \tag{4.8}$$

For any set $B \subset \mathscr{C}$, it is convenient to define

$$Ent(f, B) = \frac{1+m}{2}\int_B \psi^+ \log(\psi^+) + \frac{1-m}{2}\int_B \psi^- \log(\psi^-)$$

Note that (4.5) implies that $Ent(f, B) \geqslant 0$.

Now, after a simple algebra (2.9) reads

$$\begin{aligned} F[m+f] - F[m] &= -\frac{1}{2}(J * f, f) + \frac{1}{\beta} Ent(f, \mathscr{C}) \\ &= I_1 + I_2 + \delta Ent(f, \mathscr{C}) + (1-\delta)\, Ent(f, A) \end{aligned}$$

with, if $\tilde{f} = fI_{A^c} - \int_{A^c} f$

$$I_1 = \tfrac{1}{2} \iint J(x-y)\, \tilde{f}(x)\, \tilde{f}(y) + (1-\delta)\, Ent(f, A^c)$$

$$I_2 = -\int_A \int_{A^c} J(x-y)\, f(x)\, f(y) - \tfrac{1}{2} \int_A \int_A J(x-y)\, f(x)\, f(y) - \tfrac{1}{2} \left(\int_{A^c} f \right)^2$$

Step 3. We show that for some $C > 0$ independent of n, $I_1 \geqslant -C(\int f^2)\, Ent(f, \mathscr{C})$.

First for $x \in A^c$, $|f(x)| \leqslant \delta$. Thus, as $\delta \leqslant 2(1-m)$

$$\left| \psi^+ \log(\psi^+) - \left(\frac{f(x)}{1+m} + \frac{f(x)^2}{2!(1+m)^2} \right) \right| \leqslant \delta \frac{f(x)^2}{3!(1+m)^3}$$

and a similar inequality for ψ^-. Adding up these inequalities and recalling (4.7),

$$(1-\delta)\, Ent(f, A^c) \geqslant \frac{\phi''(\tilde{m})}{2} \int_{A^c} f^2 \geqslant \frac{\phi''(\tilde{m})}{2} \left[\int \tilde{f}^2 + \left(\int_{A^c} f_n \right)^2 \right]$$

Note that

$$\left(\int_{A^c} f \right)^2 = \left(\int_A f \right)^2 \leqslant (1+m)^2\, |A|^2$$

Thus,

$$I_1 \geqslant \frac{1}{2} \iint J(x-y)\, \tilde{f}(x)\, \tilde{f}(y) + \frac{\phi''(\tilde{m})}{2\beta} \int \tilde{f}^2 - (1+m)^2 \frac{\phi''(\tilde{m})}{2} |A|^2$$

Finally, by (4.2), (4.6) and (4.8),

$$I_1 \geqslant -(1+m)^2 \frac{\phi''(\tilde{m})}{2} |A|^2 \geqslant -C \left(\int f^2 \right) Ent(f, \mathscr{C})$$

Step 4. We show that $I_2 = o(Ent(f, \mathscr{C}))$.

We have already seen that

$$\left(\int_{A^c} f \right)^2 \leqslant (1+m)^2 \frac{\phi''(\tilde{m})}{2} |A|^2$$

Also,

$$\left|\int_A \int_A J(x-y)\, f(x)\, f(y)\right| \leqslant 4\, |J|_\infty\, |A|^2$$

Finally,

$$\left|\int_A \int_{A^c} J(x-y)\, f(x)\, f(y)\right| \leqslant |J|_\infty \int_A |f| \int_{A^c} |f| \leqslant \frac{1}{\varepsilon}\left(\int_A |f|\right)^2 + \varepsilon\left(\int |f|\right)^2$$
$$\leqslant \frac{1}{\varepsilon}\, |A|^2 + C\varepsilon Ent(f, \mathscr{C})$$

This completes the proof of Theorem 4.1. ∎

Corollary 4.2. If m is in the metastable region, there is $\varepsilon_o > 0$ and $\tilde{\gamma} > 1$ such that

$$|u-m|_2 \leqslant \varepsilon_o \Rightarrow \tilde{\gamma}\, |u-m|_2^2 \geqslant F[u] - F[m] \geqslant \frac{1}{\tilde{\gamma}}\, |u-m|_2^2 \tag{4.9}$$

Proof. By the upper bound of (4.5), it is trivial to see that for γ large, $\gamma\, |u-m|_2^2 \geqslant F[u] - F[m]$. Also, the previous proof shows that $F[u] - F[m] \geqslant \delta Ent(u-m, \mathscr{C}) + o(Ent(u-m, \mathscr{C}))$, and the two estimates of (4.5) imply that there is $\varepsilon_o > 0$ such that for γ large enough (4.9) holds. ∎

Lemma 4.3. For $\beta > 1$, and $m \in [-m^+, -\tilde{m}) \cup (\tilde{m}, m^+]$, there is $\varepsilon_o > 0$ such that if $u_o \in M_m$ and $|u_o - m|_2 < \varepsilon_o$ then, there is $C > 0$ such that

$$|u(t) - m|_2 \leqslant |u_o - m|_2\, e^{-Ct}$$

Proof. We multiply both sides of (3.1) by $u - m$ and integrate by parts

$$\frac{d}{dt}\, |u(t) - m|_2^2 + |u_x(t)|_2^2$$
$$= \beta(1-m^2) \int (J * u_x)\, u_x\, dx + \beta \int (J * u_x)\, u_x(u^2 - m^2)$$

Now, as $\int u_x = 0$, we have that

$$\int (J * u_x)\, u_x\, dx = -\tfrac{1}{2} \int [J(x-y) - 1](u_x(x) - u_x(y))^2\, dx\, dy$$

Set $\gamma = \phi''(m) - \phi''(\tilde{m}) > 0$. By the Cauchy–Schwarz inequality

$$\begin{aligned}
&\frac{d}{dt}\,|u(t)-m|_2^2 + (1-m^2)\,\gamma\,|u_x(t)|_2^2\\
&\qquad + \frac{(1-m^2)\,\beta}{2}\iint\left[J(x-y) - 1 - \frac{\phi''(\tilde{m})}{\beta}\right](u_x(x)-u_x(y))^2\\
&\leqslant \beta \int (J * u_x)\,u_x(u^2-m^2) \leqslant \beta\,|J_x * (u-m)|_\infty\,|u_x|_2\,|u-m|_2 \quad [\text{C–S}]\\
&\leqslant \beta\,|J_x|_1\,|u-m|_\infty\,|u_x|_2\,|u-m|_2 \quad [\text{Young}]\\
&\leqslant \beta\,|J_x|_1\,|u-m|_2^{3/2}\,|u_x|_2^{3/2} \quad [\text{Agmon}]\\
&\leqslant \frac{(1-m^2)\,\gamma}{2}\,|u_x(t)|_2^2 + C\,|u-m|_2^6
\end{aligned}$$

with $C = (2\gamma(1-m^2)/3)^{-3}\,(\beta\,|J_x|_1)^4$. We obtain by Poincaré's inequality and (4.2),

$$\frac{d}{dt}\,|u(t)-m|_2^2 + \frac{(1-m^2)\,\gamma}{2}\,|u(t)-m|_2^2 \leqslant C\,|u(t)-m|_2^6$$

Now, $|u(t)-m|_2$ decays as soon as

$$|u_o - m|_2^4 \leqslant \min\left(1, \frac{(1-m^2)\,\gamma}{4C}\right)$$

and the conclusion of the lemma follows easily. ∎

Remark 4.4. It is easy to see that Corollary 4.2 and Lemma 4.3 hold uniformly in $(m-\gamma, m+\gamma) \subset (\tilde{m}, m^+)$ and γ small, with the following trivial changes: there is ε_o, $\tilde{\gamma}$ and $C>0$ such that if $|u_o - \int u_o|_2 \leqslant \varepsilon_o$ and $\int u_o \in [m-\gamma, m+\gamma]$, then

$$\left|u(u_o, t) - \int u(u_o, t)\,dx\right|_2 \leqslant e^{-Ct}$$

and,

$$\tilde{\gamma}\left|u_o - \int u_o\right|_2^2 \geqslant F[u_o] - F\left[\int u_o\right] \geqslant \frac{1}{\tilde{\gamma}}\left|u_o - \int u_o\right|_2^2 \tag{4.10}$$

Remark 4.5. It is straightforward to see that if m is in the metastable region and the support of J is sufficiently small, there exists a nonconstant profile $u \in M_m$ such that $F[u] < F[m]$.

Remark 4.5 can be directly checked by taking $u(x) = m^+$ for $x \in [0, \bar{x}] \subset \mathscr{C}$ and $u(x) = -m^+$ otherwise, with $\bar{x} = (1 + (m/m^+))/2$. The fact that we have to choose J supported on a small interval is natural if we look at the problem on $L\mathscr{C}$, with $L > 1$, rather then on $\mathscr{C}$, and choose J supported in $(-1/2, 1/2) \subset L\mathscr{C}$ (by a scaling transformation we can go back to $\mathscr{C}$, where J would be supported on a interval of size $2/L$). The limit considered in refs. 9 and 17 corresponds to taking $L \to \infty$. In this rescaled setting Remark 4.5 simply says that for any m fixed in the metastable interval, we can choose L sufficiently large such that there will be a spatially nontrivial profile u with $\int_{L\mathscr{C}} u(x)\, dx = mL$ with $F[u] < F[m]$. The larger the space (i.e., L), the larger the interval of values of m for which $F[m]$ is not the global minimum in M_m and this interval will approach the whole metastable region as $L \to \infty$.

5. LARGE DEVIATIONS

In the Freidlin–Wentzell[8] approach to estimating the exit time from a domain, estimates on the probability of rare events in finite time play a fundamental role. The upper bound on the expectation of the exit time will be obtained from a lower bound for the probability of remaining in a small tube about a trajectory which performs the escape at *low cost*, such as the trajectory Φ that we will encounter in the proof of Lemma 6.4. With respect to the by now standard results on Large Deviations from the hydrodynamic limit (see e.g., ref. 14) we will need this lower bound to be uniform over configurations in an arbitrary small neighborhood of $[m - \gamma, m + \gamma]$.

On the other hand, the lower bound on the exit time follows from a Large Deviation upper bound. One difficulty in obtaining the upper bound is that our rate functional I_T, defined in 5.1 (5.2), is neither given by a supremum over l.s.c. functionals nor convex. Thus, in our approach we need to show (Lemma 5.10) that we have a *good rate function* in the terminology of ref. 7.

5.1. The Large Deviation Principle

Note that in the weak topology on M_*, a basis of neighborhoods of 0 is

$$B_{\delta, \varepsilon} = \{u : |u * \varphi_\delta|_\infty < \varepsilon\}$$

where φ_δ is an approximate identity on $\mathscr{C}$, as introduced in Section 2.

We define, see ref. 14, for $u \in D([0, T], M_*)$ and $G \in C^{1,2}([0, T] \times \mathscr{C})$

$$
\begin{aligned}
l(u, G) &= \int_0^T \int (u_t - \tfrac{1}{2}[u_x - \beta(1-u^2) J_x * u]_x)\, G\, dx\, ds \\
\Lambda_T(u, G) &= l(u, G) - \tfrac{1}{2} \int_0^T \int (1-u^2)(G_x)^2\, dx\, ds
\end{aligned}
\tag{5.1}
$$

We stress once again that we give a meaning to (5.1), and to many other formulas, by making formal integration by parts, moving in this way all the derivatives to G. For the same u, we define also the rate functional

$$
I_T(u) = \sup_{G \in C^{2,1}(\mathscr{C} \times [0, T])} \Lambda_T(u, G) \tag{5.2}
$$

In the proofs we will also need the corresponding quantity used in ref. 14 for SSEP

$$
\tilde{I}_T(u) = \sup_{G \in C^{2,1}} \left(\int_0^T \int \left(u_t - \tfrac{1}{2} u_{xx} \right) G\, dx\, dt - \tfrac{1}{2} \int_0^T \int (1-u^2)(G_x)^2\, dx\, dt \right) \tag{5.3}
$$

A partial outcome of this section is summed up in the following Large Deviation statement. Since we give it only for deterministic initial conditions our Large Deviation functional with initial condition $u_0 \in M_*$ will be

$$
\bar{I}_T(u) = \begin{cases} I_T(u) & \text{if } u(0, \cdot) = u_0(\cdot) \\ +\infty & \text{otherwise} \end{cases} \tag{5.4}
$$

Given any measurable set $A \subset D([0, T], M_*)$, we denote by $\mathring{A}$ the interior of A and by $\bar{A}$ its interior.

Theorem 5.1. For any $u_0 \in M_*$ and any $T > 0$, $\bar{I}_T$ is a good rate functional,[7] i.e., it is lower semicontinuous and it has compact level sets. If $\mu_N(0) \in M_*$ converges (weakly) to u_0, then the sequence $\{\mu_N\}_{N \in \mathbb{Z}^+}$, $\mu_N \in D([0, T]; M_*)$, has a full Large Deviation principle with rate functional $\bar{I}_T$, i.e., for any measurable set $A \subset D([0, T], M_*)$

$$
\begin{aligned}
-\inf_{u \in \mathring{A}} \bar{I}_T(u) &\leqslant \liminf_{N \to \infty} \frac{1}{N} \log(P_N^{\eta_N}(\mu_N(\cdot) \in \mathring{A})) \\
&\leqslant \limsup_{N \to \infty} \frac{1}{N} \log(P_N^{\eta_N}(\mu_N(\cdot) \in \bar{A})) \leqslant -\inf_{u \in \bar{A}} \bar{I}_T(u)
\end{aligned}
\tag{5.5}
$$

where $\eta_{N_i} = \mu_N(0, i/N)$.

Theorem 5.1 is a consequence of Corollary 5.8, Remark 5.9, Lemma 5.10 (lower semicontinuity as well as compactness of level sets of $\bar{I}_T$ follows from the same properties for I_T) and Theorem 5.11.

We recall the superexponential estimate of ref. 14, Theorem 2.1, that we will use in the form:

Proposition 5.2. Set $J=0$. For any $\delta>0$

$$\limsup_{\varepsilon\to 0}\ \limsup_{N\to\infty}\ \sup_{\eta_N\in X_N}\frac{1}{N}\log\left(P_N^{\eta_N}\left(\frac{1}{N}\int_0^t V_{N,\varepsilon}(\eta(s))\,ds\geqslant\delta\right)\right)=-\infty \tag{5.6}$$

where,

$$V_{N,\varepsilon}(\eta)=\sum_{i=1}^{N}\left|\frac{1}{N\varepsilon+1}\sum_{|i-j|\leqslant N\varepsilon}\frac{(\eta_{i+j}-\eta_i)^2}{2}-(1-(\mu_N(\eta)*\varphi_\varepsilon(i/N))^2)\right|$$

We give here also a Lemma which clarifies the relationship between $\tilde{I}_T$ and I_T.

Lemma 5.3. $\tilde{I}_T(u)<\infty$ is equivalent to $I_T(u)<\infty$. Moreover if one of the previous inequality holds, then $I_T(u)=\tilde{I}_T(u)+R_T(u)$, with

$$R_T(u)=-\frac{\beta}{2}\left[\int uJ*u\right]_0^T+\frac{\beta}{2}\int_0^T\int uJ_{xx}*u+\frac{\beta^2}{2}\int_0^T\int(1-u^2)(J_x*u)^2 \tag{5.7}$$

Proof. Let us first assume that $\tilde{I}_T(u)<\infty$. By Lemma 5.1 of ref. 14 there is $\tilde{H}\in L^2([0,T];H^1)$ such that

$$\tilde{I}_T(u)=\tfrac{1}{2}\int_0^T\int(1-u^2)(\tilde{H}_x)^2 \tag{5.8}$$

and u satisfies weakly

$$u_t=[\tfrac{1}{2}u_x-(1-u^2)\,\tilde{H}_x]_x \tag{5.9}$$

Equation (5.8) and the definition of $\tilde{I}_T$ imply that for any $G\in C^{1,2}$

$$(\tilde{l}(u,G))^2\leqslant 2\tilde{I}_T(u)\int_0^T\int(1-u^2)\,G_x^2$$

But, by the definition of l and $\tilde{l}$, we have

$$l(u, G) = \tilde{l}(u, G) - \beta \int_0^T \int (1-u^2)(J_x * u)\, G_x$$

Observe that

$$l(u, G)^2 \leqslant 2\tilde{l}(u, G)^2 + \left(\int_0^T \int (1-u^2)(\beta J_x * u)^2\right) \int_0^T \int (1-u^2)\, G_x^2$$

there is $C > 0$ such that

$$l(u, G)^2 \leqslant C \int_0^T \int (1-u^2)\, G_x^2$$

which implies that $I_T(u) < \infty$.

Assume now that $I_T(u) < \infty$. The very same argument as in Lemma 5.1 of ref. 14 implies that there exists $H \in L^2([0, T]; H^1)$ such that

$$I_T(u) = \tfrac{1}{2} \int_0^T \int (1-u^2)(H_x)^2$$

and u solves

$$u_t = [\tfrac{1}{2} u_x - (1-u^2)(\beta J_x * u + H_x)]_x$$

Therefore $\tilde{H} = H + \beta J * u$ and

$$\begin{aligned} R_T(u) &\equiv I_T(u) - \tilde{I}_T(u) \\ &= \tfrac{1}{2} \int_0^T \int (1-u^2)(\beta J_x * u)^2 - \int_0^T \int (1-u^2)\, \tilde{H}_x (\beta J_x * u)^2 \end{aligned}$$

By using (5.9) we can express $R_T(u)$ in terms of u and the proof is complete. ▮

5.2. Lower Bound and Uniform Estimates

The lower bound on the probability of rare events is achieved via a *change of measure* procedure. As in ref. 14, we study small asymmetric perturbations of our original process: with respect to this new process, unlikely events will become typical.

Therefore we introduce a process with generator

$$\Lambda_{\mathscr{H}} f(\eta) = \sum_{i=1}^{N} c_i^N(\eta)\, D^i f(\eta), \qquad \text{with} \quad c_i^N = \frac{N^2}{2} \exp\left(\frac{1}{2} D^i \mathscr{H}\right)$$

where $D^i f(\eta) = f(\eta^{i+1}) - f(\eta)$ (f is any function from X_N to $\mathbb{R}$) and,

$$\mathscr{H}(\eta) = \frac{2}{N} \sum_{k=1}^{N} \sum_{l=1}^{N} J\left(\frac{k-l}{N}\right) \eta_k \eta_l + \sum_{i=1}^{N} 2g\left(\frac{i}{N}, t\right) \eta_i$$

and $g \in C^{2,1}$ satisfies

$$\int_{\mathscr{C}} g_{xx}(t, x)^2\, dx \leqslant M^2, \qquad \text{and} \qquad \int_{\mathscr{C}} g(t, x)\, dx = 0 \qquad \forall t \leqslant T \tag{5.10}$$

The set of all such g's will be called $\mathscr{M}$. As a consequence of (5.10), $|g(t, \cdot)|_\infty \leqslant M$ and $|g_x(t, \cdot)|_\infty \leqslant M$ for any $t \leqslant T$. Also, if

$$A_i(\phi) = N \int_{(i/N,\, (i+1)/N]} \phi(y)\, dy$$

then,

$$D^i \mathscr{H}(\eta) = \frac{(\eta_i - \eta_{i+1})}{N} \left[\mu_N * J_x\left(\frac{i}{N}\right) + A_i(g_x) \right] + O\left(\frac{|J_x|_\infty}{N^2}\right) \tag{5.11}$$

and,

$$|D^i \mathscr{H}| \leqslant \frac{1}{N} (|J_x|_\infty + |g_x|_\infty) \leqslant \frac{K(M)}{N} \tag{5.12}$$

where $K(M)$ is a constant which depends only on M.

Remark 5.4. The bounds (5.11) and (5.12) show that (5.6) holds for the process with $J \neq 0$ and also for the process generated by $\Lambda_{\mathscr{H}}$. In this latter case the supremum over $\eta_N \in X_N$ can be replaced by the supremum over $\eta_N \in X_N$ and $g \in \mathscr{M}$. This follows from a direct bound on the Radon–Nikodym derivative of the law of new process between 0 and T with respect to the law of the SSEP in the same interval of time.[14]

In this subsection, we denote by $u(u_o, t)$ the weak solution at time t of the (more general) evolution equation

$$u_t = \tfrac{1}{2}[u_x - \beta(1 - u^2)(J_x * u + g_x)]_x \tag{5.13}$$

with initial condition u_o. Similarly to what was done in Lemma 3.2, if $\Psi(t, x)$ is the heat kernel in $\mathscr{C}$, $u(u_o, t)$ can be represented as

$$u(u_o, t) = \Psi(t) * u_o + \beta \int_0^t \Psi_x(t-s) * [(J_x * u)(1-u^2)(s) + g_x]\, ds \tag{5.14}$$

Now, we convolve with φ_δ both sides of Eq. (5.14) to obtain

$$\begin{aligned} &\varphi_\delta * u(u_o, t) \\ &\quad = \Psi(t) * \varphi_\delta * u_o + \beta \int_0^t \Psi_x(t-s) * \varphi_\delta * [(J_x * u)(1-u^2)(s) + g_x]\, ds \end{aligned} \tag{5.15}$$

We would like to replace $(J_x * u)(1-u^2)$ by $(J_x * u)(1-(\varphi_\delta * u)^2)$. We will use the following Lemma.

Lemma 5.5. For any $u \in M_*$ and bounded ϕ,

$$\forall \delta > 0, \quad \left| \int_0^t \int_{\mathscr{C}} (u * \varphi_\delta(s, y) - u(s, y))\, \phi(s, y)\, dy\, ds \right|$$
$$\leqslant \delta \sup_{s \leqslant t} |\phi(s)|_2 \int_0^t |u_x(s)|_2\, ds$$

Proof. We write

$$u(s, y) - u(s, y-x) = \int_0^x u_x(s, y-z)\, dz$$

and by the Cauchy–Schwarz inequality

$$\begin{aligned} &\left| \int_0^t \int_{\mathscr{C}} (u * \varphi_\delta(s, y) - u(s, y))\, \phi(s, y)\, dy\, ds \right| \\ &\quad \leqslant \left| \int_0^t ds \int dx\, \varphi_\delta(x) \int_0^x dz \left[\int \phi(s, y)\, u_x(s, y-z)\, dy \right] \right| \\ &\quad \leqslant \left(\int_{\mathscr{C}} |x|\, \varphi_\delta(x)\, dx \right) \int_0^t |u_x(s)|_2\, ds\, (\sup_{s \leqslant t} |\phi(s)|_2) \\ &\quad \leqslant \delta \sup_{s \leqslant t} |\phi(s)|_2 \int_0^t |u_x(s)|_2\, ds \quad \blacksquare \end{aligned} \tag{5.16}$$

Lemma 5.6. For $u(u_o, t)$ solution of (5.14), with any u_o, there is a constant $K(M) > 0$ depending only on M such that

$$\left| \varphi_\delta * u(u_o, t) - \Psi(t) * \varphi_\delta * u_o \right.$$
$$\left. + \beta \int_0^t \Psi_x(t-s) * \varphi_\delta * [(J_x * u)(1 - (\varphi_\delta * u)^2)(s) + g_x]\, ds \right| \leqslant \delta K(M) \tag{5.17}$$

Proof. After integrating by parts (5.13), it is easy to see that there is $\tilde{K}(M) < \infty$ such that

$$\int_0^t |u_x|_2^2\, ds \leqslant 1 + \tilde{K}(M)\, t$$

Then, we apply Lemma 5.5. ▮

Let $P^\eta_{\mathscr{H}, N}$ denotes the measure on the paths of the process generated by $\Lambda_{\mathscr{H}}$ starting at η.

Theorem 5.7. For any δ, $\varepsilon_o > 0$, there is a neighborhood B of 0 and N_o such that for any $u_o \in D$, $N > N_o$ and η with $\eta_N(\eta) \in u_o + B$, we have

$$P^\eta_{\mathscr{H}, N}(|(u(u_o, T) - \mu_N(\eta_T)) * \varphi_\delta|_\infty < \varepsilon_o) \geqslant 1 - \varepsilon_o \tag{5.18}$$

Proof. We consider

$$F_{N,\delta}(\eta, t, x) = \mu_N(\eta) * \Psi(t) * \varphi_\delta$$

where $\Psi(t)$ is the heat kernel on $\mathscr{C}$ introduced in Lemma 3.2. For a fixed t, the rate of change of $F_{N,\delta}(\eta(s), t-s, x)$ is

$$dF_{N,\delta}(\eta(s), t-s, x) = \sum_{i=1}^N c_i^N(\eta(s))\, D^i F_{N,\delta}(\eta(s), t-s, x)\, ds$$
$$- \mu_N(\eta(s)) * \Psi_t(t-s) * \varphi_\delta\, ds + dM_N(s, x) \tag{5.19}$$

where, for each $x \in \mathscr{C}$, $M_N(s, x)$ is a $P^\eta_{\mathscr{H}, N}$-martingale with

$$E^\eta_{\mathscr{H}, N} M_N(s, x)^2 = \sum_i \int_0^s E^\eta_{\mathscr{H}, N} c_i^N(\eta(s'))(D^i F_N(\eta(s'), x))^2\, ds' \leqslant \frac{sK(M)\, |\varphi'_\delta|_\infty}{N} \tag{5.20}$$

We now expand c_i^N in (5.19) and rearrange the terms

$$\begin{aligned} dM_N(s,x) &= dF_{N,\delta}(\eta(s), t-s, x) + \mu_N(\eta(s)) * (\Psi_t(t-s) * \varphi_\delta)(x) \\ &\quad - \frac{N}{2}\Bigg[\sum_{i=1}^{N} [\eta_i(s)(A_{i+1} + A_{i-1} - 2A_i)(\tau_x \Psi_x(t-s) * \varphi_\delta) \\ &\quad - \frac{(\eta_i(s) - \eta_{i+1}(s))^2}{N}(A_i - A_{i+1})(\tau_x \Psi_x(t-s) * \varphi_\delta) \\ &\quad \times \left[\mu_N * J_x\left(\frac{i}{N}\right) + A_i(g')\right]\Bigg] + t \cdot O\left(\frac{|\varphi'_\delta|_\infty \, |J_x|_\infty}{N}\right) \end{aligned}$$

Thus, if we integrate up to time t

$$\begin{aligned} M_N(t,x) &= F_{N,\delta}(\eta(t), 0, x) - F_{N,\delta}(\eta(0), t, x) \\ &\quad + \int_0^t \mu_N(\eta(s)) * (\Psi_t(t-s) + \Psi_{xx}(t-s)) * \varphi_\delta)(x)\, ds \\ &\quad - \int_0^t \Bigg[\frac{1}{2N}\sum_{i=1}^{N}\Bigg((\eta_i(s) - \eta_{i+1}(s))^2 A_i(\tau_x \Psi_x(t-s) * \varphi_\delta) \\ &\quad \times \left[\mu_N * J_x\left(\frac{i}{N}\right) + A_i(g')\right]\Bigg)\Bigg]\, ds + t \cdot O\left(\frac{|\varphi'_\delta|_\infty \, [\,|J_x|_\infty + |g'|_1]}{N}\right) \end{aligned} \tag{5.21}$$

It is important to note that for x and x' sufficiently close,

$$\sup_{\eta,\, g} |M_N(t,x) - M_N(t,x')| \leqslant K(M,\delta)(1/N + |x - x'|) \tag{5.22}$$

Now, we first focus on the term

$$\int_0^t \frac{1}{N}\sum_{i=1}^{N}\left(\frac{(\eta_i(s) - \eta_{i+1}(s))^2}{2} A_i(\tau_x \Psi_x(t-s) * \varphi_\delta)\left[\mu_N * J_x\left(\frac{i}{N}\right) + A_i(g')\right]\right) \tag{5.23}$$

We need to replace $(\eta_i(s) - \eta_{i+1}(s))^2$ by a function of the empirical density. To be able to use the superexponential estimate, we will need first to average this quantity over region of macroscopic size δ. However, when performing the required integration by parts, the test function may be smooth only on the scale δ, thus producing a non vanishing error. We therefore break the time integral into $\int_0^{t-\delta'}$ and $\int_{t-\delta'}^t$, with $\delta \ll \delta' \ll 1$, and use the regularity of Ψ_x to estimate directly the second piece $\int_{t-\delta'}^t$.

We note also that we can replace the sum in (5.23) by and integral, with an error of order $c(\delta)/N$, which is therefore negligible since we take first N to infinity. Thus, define

$$A((\eta), t, N, \delta) \equiv \left| \int_0^t \Psi_x(t-s) * \varphi_\delta * [(Q(\cdot; \eta(s)) - P(\cdot; \eta(s)))\, f(\eta(s), \cdot, s)]\, ds \right|_\infty$$

where

$$f(\eta, x, t) \equiv (\mu_N(\eta) * J_x)(x) + g_x(x, t)$$

$$Q(x; \eta) = \frac{(\eta([xN]+1) - \eta([xN]))^2}{2}$$

and

$$P(x; \eta) = 1 - (\mu_N(\eta) * \varphi_\delta)^2\,(x)$$

Now, we use twice the inequality $|g * f|_\infty \leqslant |g|_1\, |f|_\infty$ and the fact that $|\varphi_\delta|_1 = 1$ to deal with the integration from $t-\delta'$ to t

$$\left| \int_{t-\delta'}^t \Psi_x(t-s) * \varphi_\delta * [Q(\cdot; \eta(s))\, f(\eta, \cdot, s)]\, ds \right|_\infty$$

$$\leqslant \int_{t-\delta'}^t |\Psi_x(t-s)|_1\, |\varphi_\delta * [G(\cdot; \eta(s))\, f(\eta, \cdot, s)]|_\infty$$

$$\leqslant \sup_{\eta, s} \left(|Q(\eta)|_\infty\, |f(\eta, s)|_\infty \right) \int_0^{\delta'} \frac{1}{\sqrt{s}}\, ds \leqslant c\sqrt{\delta'}$$

and a similar estimate holds also with P. Thus,

$$A((\eta), t, N, \delta) \leqslant \left| \int_0^{t-\delta'} \Psi_x(t-s) * \varphi_\delta * [(Q(\cdot; \eta(s)) - P(\cdot; \eta(s)))\, f(\eta, \cdot, s)]\, ds \right|_\infty + 2c\sqrt{\delta'}$$

We need now to substitute $Q(x;\eta)$ with $\mathrm{Av}_{\delta}(Q)(x) \equiv (1/2\delta) \times \int_{|y| \leqslant \delta} Q(x+y)\,dy$. We first estimate the difference

$$\left| \int_0^{t-\delta'} \Psi_x(t-s) * \varphi_\delta * [(Q(\cdot;\eta(s))\, f(\eta,\cdot,s) - \mathrm{Av}_{\delta}(Q(\cdot;\eta(s))\, f(\eta,\cdot,s)))]\, ds \right|_\infty$$

$$= \left| \int_0^{t-\delta'} [\Psi_x(t-s) - \mathrm{Av}_{\delta}(\Psi_x(t-s))] * \varphi_\delta * (Q(\cdot;\eta(s))\, f(\eta,\cdot,s)\, ds \right|_\infty \tag{5.24}$$

Now, since $\sup_{s,\eta} |f_x|_\infty \leqslant c$, we can take f out of the average with little expense. We use now the fact (easily proven from the explicit form of the heat kernel) that for $s \geqslant \delta'$,

$$|\Psi_x(s) - \mathrm{Av}_{\delta}(\Psi_x(s))|_1 \leqslant c\, \delta/(\delta')^{5/2}$$

so that the term in (5.24) is bounded by $c'\delta/(\delta')^{5/2}$ and vanishes as $\delta/(\delta')^{5/2} \to 0$ when $\delta \to 0$.

In summary, we have seen that

$$A((\eta), t, N, \delta) \leqslant c\sqrt{\delta'} + c' \frac{\delta}{(\delta')^{5/2}} + R_{N,\delta} \tag{5.25}$$

where we have called

$$R_{N,\delta} = \sup_{s,\eta} |f|_\infty \sup_{s \in [0,t-\delta']} |\Psi_x(t-s) * \varphi_\delta|_\infty \times \int_0^{t-\delta'} |\mathrm{Av}_{\delta} Q(\cdot;\eta(s)) - P(\cdot;\eta(s))|_1\, ds \tag{5.26}$$

Finally, using that $\Psi_t + \Psi_{xx} = 0$, we obtain

$$\begin{aligned} \mu_N(\eta(t)) * \varphi_\delta(x) &= \Psi(x) * \mu_N(\eta(0)) * \varphi_\delta(x) \\ &\quad + \int_0^t [\Psi_x(t-s) * \varphi_\delta * [(1 - (\varphi_\delta * \mu_N(s))^2)] \\ &\quad \times ((\varphi_\delta * \mu_N(s)) * J_x + g_x)]\, ds + O\left(\frac{K(\delta)\, |J_x|_\infty + |g''|_2]}{N}\right) \\ &\quad + O\left(\sqrt{\delta'} + \frac{\delta}{(\delta')^{5/2}}\right) + R_{N,\varepsilon} + M_N(t,x) \end{aligned} \tag{5.27}$$

We only need to consider paths η_t such that

$$\sup_x \left[\sup_{t \leqslant T} |M_N(t, x)| + R_{N,\varepsilon}(x)\right] < \tilde{\varepsilon} \tag{5.28}$$

Indeed (5.20), (5.22), Chebychev and Doob inequalities imply that

$$\sup_\eta P^\eta_{\mathscr{H}, N}(\sup_x \sup_{t \leqslant T} |M_N(t, x)| \geqslant \tilde{\varepsilon}/2) \leqslant \frac{K(M, \delta) \sup_\eta E^\eta_{\mathscr{H}, N}[M_N(T)^2]}{\tilde{\varepsilon}^2}$$

On the other hand, the estimate (5.6), see Remark 5.4, implies that

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \sup_\eta P^\eta_{\mathscr{H}, N}(\sup_x R_{N,\varepsilon}(x) \geqslant \tilde{\varepsilon}/2) = 0$$

Now the proof is easily completed if we define $w(t) = \mu_N(\eta_t) - u(u_o, t)$ and we apply Gronwall's Lemma, as in the proof of Lemma 3.2. ∎

In the last section we will need a lower bound on the probability of being close to a specific smooth trajectory $\Phi(t)$, $t \in [0, T]$. Let $\delta_o > 0$ and $T = T(\delta_o)$ as in the proof of Lemma 6.4.

Corollary 5.8. For any δ and $\varepsilon > 0$, there is a neighborhood of 0, B, and N_o such that for $N > N_o$ and η with $\mu_N(\eta) \in m + B$,

$$\frac{1}{N} \log P^\eta_N(|(\Phi(T) - \mu_N(\eta_T)) * \varphi_\delta|_\infty < \varepsilon) \geqslant -I_T(\Phi) - \varepsilon \tag{5.29}$$

Proof. This follows repeating the standard change of measure argument, as in the proof of the lower bound in ref. 14, and by applying Theorem 5.7. ∎

Remark 5.9. To complete the proof of the lower bound in Theorem 5.1 one needs to show that $\inf_{u \in \mathring{A}} \bar{I}_T(u)$ does not change if the infimum is taken only over smooth functions. This can be proven by following for example the line of the proof of Corollary 6.5 below. Since this part of Theorem 5.1 is not used in the sequel, we omit the details.

5.3. Upper Bound

In this section we fix $T > 0$.

Lemma 5.10. Let $D([0, T], M_*)$ be equipped with the Skorokhod topology, $M > 0$, and C closed in $D([0, T], M_*)$.

(i) $\{u \in D([0, T], M_*): I_T(u) \leqslant M\}$ is compact

(ii) $I_T: D([0, T], M_*) \to [0, \infty]$ is lower semicontinuous

(iii) $\liminf_{\varepsilon \to 0} \inf_{u \in C} I_T(\varphi_\varepsilon * u) \geqslant \inf_{u \in C} I_T(u)$

Proof.

Step 1. We first show that $\{u \in M_* : \tilde{I}_T(u) \leqslant M\}$ is compact (this was neither shown nor needed in ref. 14).

Let $\{u^n(t, x)\}$ be such that $\tilde{I}_T(u^n) \leqslant M$. There is $\{\tilde{H}^n(t, x)\}$ (see ref. 14) and $\{u_o^n(x) \in [-1, 1]\}$ such that

$$\partial_t u^n = \tfrac{1}{2} \partial_x [\partial_x u^n - (1 - (u^n)^2) \partial_x \tilde{H}^n] \tag{5.30}$$

and,

$$\tilde{I}_T(u^n) = \tfrac{1}{2} \int_0^T \int (\tilde{H}_x^n)^2 (1 - (u^n)^2) \leqslant M$$

and $u^n(0, x) = u_o^n(x)$. After multiplying (5.30) by u^n and integrating by parts over $\mathscr{C}$,

$$\int_0^T \int (u_x^n)^2 \leqslant \int_{\mathscr{C}} (u_o^n)^2 + \int_0^T \int (1 - (u^n)^2) u_x^n \tilde{H}_x^n$$

By the Cauchy–Schwarz inequality and the fact that $\mathscr{C}$ is bounded and $u_o^n \in [-1, 1]$,

$$\tfrac{1}{2} \int_0^T \int (u_x^n)^2 \leqslant 1 + \tfrac{1}{2} \int_0^T \int (1 - (u^n)^2)(\tilde{H}_x^n)^2 \leqslant 1 + M \tag{5.31}$$

First note that $\iint (u^n)^2 \leqslant T$ implies that there is $u(t, x) \in M_*$ and a subsequence of u^n (that we still call u^n here), such that

$$\forall G(t, x) \in L^2([0, T] \times \mathscr{C}, dt\, dx) \lim_{n \to \infty} \int_0^T \int G u^n = \int_0^T \int G u \tag{5.32}$$

For a smooth $G(x)$, let

$$V_n(t) = \int_{\mathscr{C}} G(x)\, u^n(t, x)\, dx$$

Then,

$$|V_n(t) - V_n(s)| = \left| \int_{\mathscr{C}} \int_s^t G(x)\, \partial_t u^n \right| = \left| \int_{\mathscr{C}} \int_s^t G_x(u_x^n + (1 - (u^n)^2)\, \tilde{H}_x^n \right|$$

$$\leqslant \sqrt{t-s}\, |G_x|_2 \left[\int_0^T \int (u_x^n)^2 + (1 - (u^n)^2)(\tilde{H}_x^n)^2 \right]^{1/2} \tag{5.33}$$

Thus, $\{V_n(t)\}$ is an equicontinuous and equibounded family. By Ascoli–Arzelà, for each G, there is $V(t, G)$ continuous and a subsequence which depends on G such that

$$\lim_{k \to \infty} \sup_{t \leqslant T} |V_{n_k}(t) - V(t, G)| = 0 \tag{5.34}$$

Because of (5.32), this means that for $H(t) \in L^2([0, T], dt)$

$$\int H(s)\, V(s, G)\, ds = \int_0^T \int u(x, s)\, G(x)\, H(s)\, dx\, ds$$

which means that we can choose u so that $\int u(s)\, G$ is continuous, and

$$\lim_{n \to \infty} \sup_{t \leqslant T} \left| \int_{\mathscr{C}} (u^n(t, x) - u(t, x))\, G(x)\, dx \right| = 0 \tag{5.35}$$

A fortiori u^n converges to u in the Skorokhod topology. Now, since $\tilde{I}_T$ is l.s.c.

$$\tilde{I}_T(u) \leqslant \liminf \tilde{I}_T(u^n) \leqslant M$$

Step 2. Let $\{u^n\}$ be such that $I_T(u^n) \leqslant M$. We show that there is a subsequence converging to $u \in M_*$ and

$$\liminf_{n \to \infty} I_T(u^n) \geqslant I_T(u)$$

First, there is $\tilde{M}$ such that $\tilde{I}_T(u^n) \leqslant \tilde{M}$ for any n. By Step 1, we know that there is a subsequence converging to u and $\tilde{I}_T(u) \leqslant \tilde{M}$. We have seen in Lemma 5.3 that $I_T(u) = \tilde{I}_T(u) + R_T(u)$. We first show that

$$\lim_{n \to \infty} \int_0^T \int (u^n)^2 = \int_0^T \int u^2 \tag{5.36}$$

It is then easy to see that $\lim R_T(u^n) = R_T(u)$, and we leave it to the reader.

Let $I_i = [(i-1)/N, i/N)$ and x_i an arbitrary point in I_i for $i = 1,..., N$. For any $t \leqslant T$,

$$\begin{aligned}\int_{\mathscr{C}} u^k(t, x)(u^k(t, x) - u(t, x))\, dx &= \sum_{i=1}^{N} \left[u^k(t, x_i) \int_{I_i} (u^k(t, x) - u(t, x))\, dx \right] \\ &\quad + \sum_{i=1}^{N} \left[\int_{I_i} (u^k(t, x) - u^k(t, x_i))(u^k(t, x) - u(t, x))\, dx \right]\end{aligned}$$

To estimate the second sum, we write for any $x \in I_i$,

$$|u^k(t, x) - u^k(t, x_i)| = \left| \int_{\mathscr{C}} I_{[x_i, x]}(y)\, u_x^k(y)\, dy \right| \leqslant \sqrt{|x_i - x| \int_{\mathscr{C}} (u_x^k)^2}$$

and in the last step we used the Cauchy–Schwarz inequality. Now, for $x \in I_i$, $|x_i - x| \leqslant 1/N$ so that the second sum has the estimate

$$\begin{aligned}&\left| \sum_{i=1}^{N} \int_{I_i} (u^k(t, x) - u^k(t, x_i))(u^k(t, x) - u(t, x))\, dx \right| \\ &\qquad \leqslant \sum_{i=1}^{N} \left[\frac{1}{N} \int_{\mathscr{C}} (u_x^k)^2 \right]^{1/2} \int_{I_i} |u^k(t, x) - u(t, x)|\, dx \\ &\qquad \leqslant \left[\frac{1}{N} \int_{\mathscr{C}} (u_x^k)^2 \right]^{1/2} \int_{\mathscr{C}} |u^k(t, x) - u(t, x)|\, dx \qquad (5.37)\end{aligned}$$

After integrating over time (5.37), and using the Cauchy–Schwarz inequality

$$\begin{aligned}&\left| \int_0^T \int u^k(t, x)(u^k(t, x) - u(t, x))\, dx\, dt \right| \\ &\qquad \leqslant NT \max_{i \leqslant N} \sup_{t \leqslant T} \left| \int_{I_i} (u^k(t, x) - u(t, x))\, dx \right| + 2 \sqrt{\frac{T}{N}} \left[\int_0^T \int (u_x^k)^2 \right]^{1/2}\end{aligned}$$

Now, just as in (5.31), $\iint (u_x^k)^2 \leqslant 1 + \tilde{M}$. Thus, for any ε, there is N independent of k such that

$$4 \frac{T}{N} \left[\int_0^T \int (u_x^k)^2 \right] \leqslant \varepsilon^2$$

Once N is fixed, by Step 1, there is k_o such that for $k > k_o$,

$$NT \max_{i \leqslant N} \sup_{t \leqslant T} \left| \int_{I_i} (u^k(t) - u(t)) \right| < \varepsilon$$

and (5.36) follows, It is easy to conclude that

$$\liminf_{n \to \infty} I_T(u^n) \geqslant I_T(u)$$

Note that this proves that I_T is l.s.c. on the compact sets $\{u : I_T(u) \leqslant M\}$ for $M > 0$. Now, to complete the proof of the l.s.c. of I_T, assume that $I_T(u) = \infty$ and that $\{u^n\}$ converges to u. From what we saw, no subsequence can belong to $\{u : I_T(u) \leqslant M\}$ for some finite M. Thus, any limit point of $\{I_T(u^n)\}$ must be infinite.

(iii) Assume the left hand side of the expression in the point (iii) of Lemma 5.10 is finite. There is $M > 0$ and for any $\varepsilon > 0$ there is $u_\varepsilon \in C$ such that $I_T(\varphi_\varepsilon * u_\varepsilon) \leqslant M$. Thus $\{\varphi_\varepsilon * u_\varepsilon\}_{\varepsilon > 0}$ belongs to a compact set and there is ε_n going to 0, $u_n \in C$ and u with $I_T(u) \leqslant M$ such that $\varphi_{\varepsilon_n} * u_n$ converges to u. Now, for any open neighborhood of 0, say B_w,

$$\exists n_o : \forall n \geqslant n_o, \qquad \varphi_{\varepsilon_n} * u_n \in B_w + C$$

Thus,

$$u \in \bigcap_{B_w} (C + \bar{B}_w) = C$$

By l.s.c. of I_T,

$$\liminf_{\varepsilon \to 0} I_T(\varphi_\varepsilon * u_\varepsilon) \geqslant I_T(u) \geqslant \inf_{u \in C} I_T(u) \quad \blacksquare$$

We show an upper bound estimate for closed set by comparison with SSEP, where this has been shown in ref. 14.

Theorem 5.11. For C closed subset of $D([0, T], M_*)$,

$$\limsup_{N \to \infty} \sup_{\eta \in X_N} \frac{1}{N} \log(P^\eta_M(\mu_N(\cdot) \in C)) \leqslant -\inf_{u \in C} I_T(u) \qquad (5.38)$$

Proof. Recall (Lemma 5.3) that $\tilde{I}_T(u) < \infty$ implies that $\tilde{I}_T(u) = I_T(u) + R_T(u)$ and we can write

$$R_T(u) = \int_0^T \int (1-u^2)\, \tilde{H}_x(J_x * u) - \tfrac{1}{2} \int_0^T \int (1-u^2)(J_x * u)^2$$

Thus, there is a constant M such that

$$R_T(u) \leqslant \tfrac{1}{3}\tilde{I}_T(u) + MT \tag{5.39}$$

For ease of reading we recall here three facts

(i) $\tilde{I}_T(\varphi_\varepsilon * u) \leqslant \tilde{I}_T(u)$ by convexity

(ii) $\limsup_{N\to\infty} \sup_\eta \dfrac{1}{N} \log(\tilde{P}^\eta_N(\mu_N(\cdot) \in C))$

$\leqslant -\inf_{u\in C} \tilde{I}_T(u)$ by Theorem 4.1 of ref. 14

(iii) $\liminf_{\varepsilon\to 0} \inf_{u\in C} I_T(\varphi_\varepsilon * u) \geqslant \inf_{u\in C} I_T(u)$ by Lemma 5.10

By Varadhan's Theorem,[7] (ii) implies that for any continuous and bounded functional F on M_*,

$$\limsup_{N\to\infty} \sup_\eta \frac{1}{N} \log(\tilde{E}^\eta_N e^{NF(\mu_N)}) \leqslant \sup_{u\in M_*} [F(u) - \tilde{I}_T(u)] \tag{5.40}$$

Note that $F_\varepsilon: u \mapsto I_T(\varphi_\varepsilon * u) \wedge M$ is continuous and bounded. We claim that

$$\limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \sup_\eta \frac{1}{N} \log E^\eta_N \exp(NF_\varepsilon)$$

$$\leqslant \limsup_{\delta\to 0} \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \sup_\eta \frac{1}{(1+\delta)N} \log \tilde{E}^\eta_N e^{(1+\delta)N[F_\varepsilon(\mu_N) + R_T(\varphi_\varepsilon * \mu_N)]} \tag{5.41}$$

Assume that for a moment. Then $F_\varepsilon(u) + F_G(\varphi_\varepsilon * u)$ is continuous and bounded and (5.40) implies that

$$\begin{aligned}
&\limsup_{N\to\infty}\sup_{\eta}\frac{1}{N}\log \tilde{E}^{\eta}_N e^{(1+\delta)N[F_\varepsilon + R_T(\varphi_\varepsilon * \cdot)]}\\
&\quad\leqslant \sup_{\tilde{I}_T(u)<\infty}[(1+\delta)(F_\varepsilon(u)+R_T(\varphi_\varepsilon * u))-\tilde{I}_T(u)]\\
&\quad\leqslant \sup_{\tilde{I}_T(u)<2M}[\delta M+\delta R_T(\varphi_\varepsilon * u)]\\
&\qquad+\sup_{\tilde{I}_T(u)\geqslant 2M}\left[2\delta M(1+T)+\frac{2\delta}{3}\tilde{I}_T(\varphi_\varepsilon * u))-\delta\tilde{I}_T(\varphi_\varepsilon * u)\right]\\
&\quad\leqslant 3\,\delta M(1+T)
\end{aligned}$$

Thus, by Chebychev inequality,

$$\limsup_{N\to\infty}\sup_{\eta}\frac{1}{N}\log P^{\eta}_N(C)\leqslant -\limsup_{\varepsilon\to 0}\inf_C I_T(\varphi_\varepsilon * u)\wedge M$$

However, (iii) implies that

$$-\limsup_{\varepsilon\to 0}\inf_C I_T(\varphi_\varepsilon * u)\wedge M\leqslant -\inf_C I_T\wedge M$$

The result would then follow as M goes to infinity. We prove now the claim (5.41).

Recall that

$$\begin{aligned}
\psi_i(\eta)&=\frac{1}{N}D^i\sum_{k,l}J\left(\frac{k-l}{N}\right)\eta_k\eta_l\\
&=(\eta_i-\eta_{i+1})\left((\mu_N * J)\left(\frac{1+i}{N}\right)-(\mu_N * J)\left(\frac{i}{N}\right)+\frac{1}{N}(J(1/N)-J(0))\right)
\end{aligned}$$

Thus

$$\begin{aligned}
Z&=\log\left(\frac{d\tilde{P}^{\eta}_N}{dP^{\eta}_N\,|\,\mathscr{F}^{\,t}_0}\right)=\sum_i\int_0^t\left[\psi_i\,dJ^i(s)+\frac{N^2}{2}(e^{\psi_i}-1)\,ds\right]\\
&=\sum_{i=1}^{N}\int_0^t\left[(\mu_N * J)\left(\frac{i}{N},s\right)d\eta_i(s)\right]\\
&\quad-\frac{N^2}{2}(\eta_i-\eta_{i+1})[-\nabla_i\mu_N * J+(J(1/N)-J(0))/N]\\
&\quad-\frac{N^2}{2}(\eta_i-\eta_{i+1})^2\left[-\nabla_i\mu_N * J+O\left(\frac{1}{N^2}\right)\right]^2
\end{aligned}$$

$$= N\left[\int_0^t (\mu_N * J, d\mu_N(s)) - \frac{1}{2}(N^2 \Delta_i(\mu_N * J), \mu_N(s))\, ds\right]$$

$$-\frac{N}{2}\sum_{i=1}^{N}\int_0^t \left[(\eta_i - \eta_{i+1})^2\left[(\nabla_i \mu_N * J)^2 + O\left(\frac{1}{N^3}\right)\right]\right]$$

$$= N\left[\int_0^t (\mu_N * J, d\mu_N(s)) - \frac{1}{2}(J_{xx} * \mu_N, \mu_N(s))\, ds\right.$$

$$\left. -\frac{1}{2}\int_0^t (f(\varphi_\varepsilon * \mu_N), (\mu_N * J_x)^2)\, ds + O\left(\frac{1}{N}\right)\right]$$

$$+\int_0^t V(N, \varepsilon, (\mu_N * J_x)^2, \eta_s)\, ds$$

$$= N[R_T(\varphi_\varepsilon * \mu_N) + O(\varepsilon) + O(1/N)] + \int_0^t V(N, \varepsilon, (\mu_N * J_x)^2, \eta_s)\, ds$$

By Hôlder's inequality

$$\log E_N^\eta e^{NF_\varepsilon} \leqslant \frac{1}{1+\delta}\log \tilde{E}_N^\eta \exp((1+\delta)\, N[F_\varepsilon(\mu_N) + R_T(\varphi_\varepsilon * \mu_N)])$$

$$+\frac{\delta}{1+\delta}\log \tilde{E}_N^\eta e^{(1+d)/\delta \int_0^t V(N, \varepsilon, (\mu_N * J_x)^2, \eta_s)\, ds} + O(\varepsilon) + O(1/N)$$

Since we take the limit in N first and then in ε, the last three terms vanish because of the superexponential estimate. ∎

6. THE QUASI-POTENTIAL

6.1. Introduction

Let m and $\gamma > 0$ be such that $[m-\gamma, m+\gamma] \subset (\tilde{m}, m^+)$. Our first goal is to build a closed neighborhood of m, say D_γ, invariant under the evolution (3.1). Though we will not prove that a random pertubation of the macroscopic equation exits D_γ at the points of ∂D_γ minimizing the energy, this idea will guide us: let ξ^* be such a minimum; we need to show also that for γ small, there are points in D_γ^c arbitrarily close to ξ^* with an energy arbitrarily close to $F(\xi^*)$. Closeness is measured in the weak topology, and the matter is not trivial since F is only l.s.c.. We have not been able to see that this was always true, rather we build in Corollary 6.3 one such D_γ based on soft arguments and the properties of (3.1).

The second contribution of this section is to give in Lemma 6.4 an explicit formula for the minimum of the rate functional over paths with initial condition m and final state, say $\xi \in D_\gamma$.

We consider the compact subspace of M_*, $S_\gamma = M_* \cap \{u : \int u \in [m-\gamma, m+\gamma]\}$. We endow S_γ with the inherited weak topology. Our idea is to work first in $L^2(S_\gamma, dx)$ and look for arbitrary small (in L^2) invariant neighborhood of $[m-\gamma, m+\gamma]$. For ease of reading we recall some useful facts. For ε_o small enough,

$$\tilde{U} = \left\{ u \in S_\gamma : \left| u - \int u \right|_2 \leqslant \varepsilon_o \right\}$$

is an L^2 closed neighborhood of $[m-\gamma, m+\gamma]$ such that

$$(*) \quad \exists C > 0, \forall u_o \in \tilde{U}, \left| u(u_o, t) - \int u_o \right|_2 \leqslant e^{-Ct} \qquad \text{[Remark 4.4]}$$

$$(**) \quad \exists \tilde{\gamma} > 1, \forall u \in \tilde{U}, \tilde{\gamma} \left| u - \int u \right|_2^2 \geqslant F[u] - F\left[\int u \right] \geqslant \frac{1}{\tilde{\gamma}} \left| u - \int u \right|_2^2$$

$$(***) \quad \forall u_o \in \tilde{U}, \forall t > 0, \frac{dF[u(u_o, t)]}{dt} < 0 \qquad \text{[Corollary 3.7]}$$

Let $\partial_2 \tilde{U}$ be the L^2 boundary of $\tilde{U}$ in S_γ. We can choose $\delta_o > 0$ such that

$$\delta_o < \inf_{v \in \partial_2 \tilde{U}} \left(F[v] - F\left[\int v \right] \right)$$

For $\delta \leqslant \delta_o$, we define

$$U_\delta = \left\{ u \in \tilde{U} : F[u] - F\left[\int u \right] \leqslant \delta \right\}, \qquad \text{and} \qquad U = U_{\delta_o}$$

The reason to choose δ_o so small is that for $u \in \partial_2 U$, $F[u] - F[\int u] = \delta_o$ (as one trivially checks), and thus once a trajectory hits $\partial_2 U$, it stays in U by $(***)$. Finally, for $T > 0$, define

$$\Gamma_{T,\gamma} = \{ u_o \in S_\gamma : \exists t \leqslant T, u(u_o, t) \in U \} \tag{6.1}$$

and, if $\partial_m \Gamma_{T,\gamma}$ denotes the weak boundary of $\Gamma_{T,\gamma} \cap M_m$ in M_m, we define

$$\alpha_T = \inf_{\partial_m \Gamma_{T,\gamma}} F[u]$$

It is clear that α_T is independent of γ. We first gather some simple properties of $\Gamma_{T,\gamma}$ and α_T.

Lemma 6.1. For $T>0$, (i) $\Gamma_{T,\gamma}$ is closed and invariant; (ii) for $\delta>0$, there is a weak neighborhood of 0, B, such that $\Gamma_{T,\gamma}+B\cap\partial\Gamma_{T+\delta,\gamma}=\varnothing$ and the weak interior of $\Gamma_{T,\gamma}$ is not empty; (iii) α_T is increasing.

Proof. Time invariance follows by definition: if $u(u_o, t)\in U$ for some $t\leqslant T$, then $\forall s\geqslant 0$, $u(u_o, t+s)\in U$, i.e., $u(u_o, s)\in\Gamma_{T,\gamma}$ for $s\geqslant 0$.

Now, we show that $\Gamma^c_{T,\gamma}$ is open. Let $u_o\in\Gamma^c_{T,\gamma}$ and $\{u^n_o\}$ converge to u_o. By definition $u(u_o, T)\in U^c$ and there is an L^2-neighborhood of 0, say B_2, such that

$$(u(u_o, T)+B_2)\bigcap U=\varnothing$$

Thus by Lemma 3.2, there is a weak open, B_w with

$$\forall v_o\in(u_o+B_w),\qquad u(v_o, T)\in(u(u_o, T)+B_2)$$

Therefore, for n large enough $u(u^n_o, T)\in u(u_o, T)+B_2$. Now, by definition of U,

$$u(u^n_o, T)\notin U\Rightarrow\forall t\leqslant T, u(u^n_o, t)\notin U$$

Thus, $u^n_o\notin\Gamma_{T,\gamma}$ for n large enough, and $\Gamma^c_{T,\gamma}$ is open.

(ii) $\partial\Gamma_{T+\delta,\gamma}$ and $\Gamma_{T,\gamma}$ being compact, we only need to show that $\partial\Gamma_{T+\delta,\gamma}\cap\Gamma_{T,\gamma}=\varnothing$. We claim that $u_o\in\partial\Gamma_{T+\delta,\gamma}$ implies that

$$u(u_o, T+\delta)\in U\qquad\text{but}\quad u(u_o, T+\delta')\notin U\qquad\text{for}\quad\delta'<\delta$$

The first part is implied by (i). For the second, assume that $u(u_o, T+\delta')\in U$ for $\delta'<\delta$. Then by $(***)$, for any $\varepsilon>0$ $u(u_o, T+\delta'+\varepsilon)\in\operatorname{int}_2 U$ so that by Lemma 3.2, there is a neighborhood around u_o where all points share the former property. This implies that $u_o\in\operatorname{int}_w\Gamma_{T+\delta,\gamma}$. Now, note that for any open B_w, $\Gamma_{T,\gamma}+B_w$ is open and what we have just seen there is B_w such that $\Gamma_{T,\gamma}+B_w\subset\Gamma_{T+\delta,\gamma}$. this implies that the interior of $\Gamma_{T,\gamma}$ is not empty.

(iii) Let $T'>T$. Recall that F is l.s.c. and $\partial_m\Gamma_{T',\gamma}$ is compact. Thus,

$$\exists u_{T'}\in\partial_m\Gamma_{T',\gamma}:\quad F[u_{T'}]=\alpha(T')\tag{6.2}$$

Now, $u_{T'} \in \partial_m \Gamma_{T',\gamma}$ means that there are $u^n \in \Gamma^c_{T',\gamma} \cap M_m$ converging to $u_{T'}$. Because $\Gamma_{T,\gamma} \cap M_m \subset \Gamma_{T',\gamma} \cap M_m$, $u_{T'} \in \Gamma_{T,\gamma}$, implies that $u_{T'} \in \partial_m \Gamma_{T,\gamma}$, but then (ii) is true

$$\alpha(T) = \inf_{\partial_m \Gamma_{T,\gamma}} F \leqslant F[u_{T'}] = \alpha(T')$$

Thus, we can assume $u_{T'} \notin \Gamma_{T,\gamma}$. By continuity of the time evolution in L^2,

$$\exists t_o \in (T, T'] \colon \quad \forall \varepsilon > 0, \quad u(u_{T'}, t_o - \varepsilon) \notin U \quad \text{and} \quad u(u_{T'}, t_o) \in U$$

In other words $u(u_{T'}, t_o - T) \in \partial_m \Gamma_{T,\gamma}$. Now, F decays along the evolution so that

$$\alpha(T) = \inf_{\partial_m \Gamma_{T,\gamma}} F \leqslant F[u(u_{T'}, t_o)] \leqslant F[u_{T'}] = \alpha(T') \quad \blacksquare$$

Lemma 6.2. For $\varepsilon > 0$, there is $\gamma_o > 0$ such that for any $\gamma < \gamma_o$

$$\inf_{\partial \Gamma_{T,\gamma}} \left(F[u] - F\left[\int u \right] \right) \geqslant \inf_{\partial_m \Gamma_{T,\gamma}} (F[u] - F[m]) - \varepsilon$$

Proof. By contradiction, suppose that there is $h > 0$, $\gamma_n \to 0$ and $u_n \in \partial \Gamma_{T,\gamma_n}$ such that

$$F[u_n] - F\left[\int u_n \right] \leqslant \inf_{\partial_m \Gamma_{T,\gamma}} (F[u] - F[m]) - h \tag{6.3}$$

There is a subsequence, say again $\{u_n\}$, and $u^* \in M_m$ so that $\{u_n\}$ converges to u^* weakly. Now, for any $\delta > 0$, $u(u_n, T+\delta) \in \operatorname{int} U$ and $u(u_n, T-\delta) \in U^c$. Lemma 3.2 implies that $u(u^*, T+\delta) \in \operatorname{int} U$ and $u(u^*, T-\delta) \in U^c$. Thus $u^* \in \partial_m \Gamma_{T,\gamma}$. The l.s.c. of F, together with the continuity of $m \mapsto F[m]$ imply that

$$\liminf F[u_n] - F\left[\int u_n \right] \geqslant F[u^*] - F[m]$$

and this contradicts (6.3). $\blacksquare$

Corollary 6.3. Let $T > 0$ be a continuity point of α. There is $\xi^* \in \partial_m \Gamma_{T,\gamma}$ such that for any $\varepsilon > 0$, there is γ_o such that for $\gamma > \gamma_o$, and any B_w, weak neighborhood of 0, there is $\zeta_\varepsilon \in (\xi^* + B_w) \cap \Gamma^c_{T,\gamma}$ and

$$\max(F[\zeta_\varepsilon] - F[m], F[\xi^*] - F[m]) \leqslant \inf_{\partial \Gamma_{T,\gamma}} \left(F[u] - F\left[\int u \right] \right) - \varepsilon \tag{6.4}$$

Proof. α is increasing, so the set continuity points is dense. Let $T>0$ be one of them. There is $\delta>0$ such that $\alpha(T+\delta)<\alpha(T)+\varepsilon/2$. Now, with the notation of (6.2) $u_{T+\delta}\notin\Gamma_{T,\gamma}$, otherwise, $u(u_{T+\delta},T)\in\partial_2 U$. This in turn would mean that $F[u(u_{T+\delta},T))-F[m]=\delta$, and because F decreases strictly in any positive time, $u(u_{T+\delta},T+\delta)$ would belong to the L^2 interior of U, contradicting $u_{T+\delta}\in\partial\Gamma_{T+\delta}$. Finally, let $\xi_\varepsilon=u(u_{T+\delta},\delta)\in\partial_m\Gamma_{T,\gamma}$. Because $\partial_m\Gamma_{T,\gamma}$ is compact, let ξ^* be a limit point of ξ_ε along a subsequence and $\zeta_\varepsilon=u_{T+\delta}$. It is easy to conclude now by recalling Lemma 6.2. ∎

We can now define D_γ as $\Gamma_{T,\gamma}$ for a continuity point of α. Also, for ξ^* as in Corollary 6.3, we can define

$$H(T)=\{u\in D([0,T],M_*):u(0,x)=m,\,u(T,x)=\xi^*(x)\}$$

Lemma 6.4.

$$F[\xi^*]-F[m]=\inf_{T>0}\inf_{H(T)}I_T(u) \tag{6.5}$$

Proof.

Step 1. We show that $I_T(u)\geqslant F[\xi^*]-F[m]$.

We may assume that $I_T(u)<\infty$. Recall that this implies that u is weakly continuous in time. We need to regularize u. Recall that for SSEP, the hydrodynamic limit of the density evolves according to the heat equation. Let P_t be the heat semi-group. For $\delta>0$, consider the extension of u on $[-1,T+1]$

$$\tilde u(x,s)\equiv m \qquad \text{for} \quad s\in[-1,0]$$

and

$$\tilde u(x,s+T)=P_s(\xi^*)(x) \qquad \text{for} \quad s\in[0,1]$$

Let $\tau^\delta u(x,t)=u(x-\delta_1,t-\delta_2)$. Because $\tilde u$ evolves according to the heat equation in $[-1,0]\cup[T,T+1]$ we can define $\tilde H$ as

$$u_t=[\tfrac12 u_x-\tilde H_x(1-u^2)]_x \tag{6.6}$$

Now, for $|\delta_2|\leqslant\varepsilon_o$ and $0<\varepsilon_o<1/2$

$$\tilde I_{-\varepsilon_o,T}(\tau^\delta\tilde u)=\int_{-\delta_2-\varepsilon_o}^{T-\delta_2}\int(\tilde H_x)^2(1-\tilde u^2)=\int_{(-\delta_2-\varepsilon_o)\vee 0}^{(T-\delta_2)\wedge T}\int(\tilde H_x)^2(1-\tilde u^2)\leqslant\tilde I_T(u)$$

Now, be convexity of $\tilde{I}_T$

$$\tilde{I}_{-\varepsilon_o, T}\left(\int_{[-1,1]^2} \tau^\delta \tilde{u}(\cdot)\, \varphi_\varepsilon(\delta_1)\, \varphi_\varepsilon(\delta_2)\, d\delta_1\, d\delta_2\right)$$

$$\leqslant \int_{[-1,1]^2} \tilde{I}_{-\varepsilon_o, T}(\tau^\delta \tilde{u})\, \varphi_\varepsilon(\delta_1)\, \varphi_\varepsilon(\delta_2)\, d\delta_1\, d\delta_2 \leqslant \tilde{I}_T(u) \tag{6.7}$$

If we denote by v_ε the time and space convolution appearing in the left-hand side of (6.7) and recall that u is weakly continuous in time, then as ε goes to 0, v_ε goes to $\tilde{u}$ in the Skorokhod topology and thus by l.s.c. $\tilde{I}_{-\varepsilon_o, T}(v_\varepsilon) \to \tilde{I}_{-\varepsilon_o, T}(\tilde{u}) = \tilde{I}_T(\tilde{u})$. This implies that $\tilde{I}_T(v_\varepsilon) \to \tilde{I}_T(u)$ and

$$\lim_{\varepsilon \to 0} I_T(v_\varepsilon) = I_T(u) \tag{6.8}$$

Now for a smooth $u(t, x)$ with $I_T(u) < \infty$, there is $H = \tilde{H} - J_x * u$ and

$$I_T(u) = \tfrac{1}{2} \int_0^T \int_{\mathscr{C}} [(H_x)^2](1 - u^2)$$

When u is smooth, the following calculation make sense

$$\begin{aligned}
&\frac{1}{2} \int_0^T \int \left[(H_x)^2 - \left(H_x - \frac{\delta F}{\delta u}[u]_x\right)^2\right] (1 - u^2) \\
&= \int_0^T \int \left[\left(H_x - \frac{1}{2} \frac{\delta F}{\delta u}[u]_x\right)\left(\frac{\delta F}{\delta u}[u]_x\right)\right] (1 - u^2) \\
&= \int_0^T \int \left[\left(\frac{1}{2} \frac{\delta F}{\delta u}[u]_x - H_x\right)(1 - u^2)\right]_x \frac{\delta F}{\delta u}[u] = \int_0^T \int u_t \frac{\delta F}{\delta u}[u] \\
&= F[u(T, \cdot)] - F[u(0, \cdot)]
\end{aligned}$$

And thus,

$$I_T(u) = \tfrac{1}{2} \int_0^T \int [(H_x)^2](1 - u^2)\, dx\, ds \geqslant F[u(T, \cdot)] - F[u(0, \cdot)]$$

In our case, the smooth function is v_ε. After integrating between $-\varepsilon_o$ and T

$$F[v_\varepsilon(T, \cdot)] - F[v_\varepsilon(-\varepsilon_o, \cdot)] \leqslant I_{-\varepsilon_o, T}(v_\varepsilon)$$

When ε is small enough, $v_\varepsilon(-\varepsilon_o, \cdot) \equiv m$. By l.s.c. of F,

$$\forall u \in H(T), \quad I_T(u) = \limsup_{\varepsilon \to 0} I_{-\varepsilon, T}(v_\varepsilon)$$
$$\geqslant F[u(T, \cdot)] - F[m] = F[\xi^*] - F[m] \tag{6.9}$$

Step 2. We show that $F[\xi^*] - F[m] \geqslant \inf_{T>0} \inf_{H(T)} I_T(u)$. In fact, consider the time reversed evolution with $u(0, x) = \xi^*(x)$ and

$$u_t = \left[\frac{1}{2}\frac{\delta F}{\delta u}[u]_x (1 - u^2)\right]_x \tag{6.10}$$

Then, by Lemma 3.3 and 4.3, for any $\delta > 0$, there is $T > 0$ and $C(1, T)$ such that

$$|u(T) - m|_\infty < \delta, \qquad \text{and} \qquad |u_x(T)|_2 \leqslant C(1, T)$$

Let $\tilde{u} = u(T)$, and for $t \in [0, \delta]$ define

$$v(t, x) = \frac{t}{\delta} m + \left(1 - \frac{t}{\delta}\right) \tilde{u}(x)$$

We need to estimate the rate function of such a density between $[0, \delta]$. Assume there is H of mean 0 such that

$$v_t = \left[(1 - v^2)\left[\frac{1}{2}\frac{\delta F}{\delta u}[v] - H\right]_x\right]_x$$

If we multiply both sides by H, use that $v_t = (m - \tilde{u})/\delta$ and integrate

$$\frac{1}{\delta}\int (m - \tilde{u})\, H\, dx = -\frac{1}{2}\int (1 - v^2)\left[\frac{\delta F}{\delta u}[v] - 2H\right]_x H_x$$

Hence, at any time $t \in [0, \delta]$

$$\int_{\mathscr{C}} (1 - v^2)\, H_x^2\, dx = \frac{1}{2}\int (1 - v^2)\frac{\delta F}{\delta u}[v]_x \cdot H_x\, dx + \frac{1}{\delta}\int (m - \tilde{u})(H)\, dx$$
$$\leqslant \frac{2}{2.4}\int (1 - v^2)\, H_x^2\, dx + \frac{2}{4}\int (1 - v^2)\left(\frac{\delta F}{\delta u}[v]_x\right)^2 dx$$
$$+ \frac{4}{2\delta^2}\left[\sup_x \frac{1}{(1 - v^2(x, t))}\right]\int (m - \tilde{u})^2\, dx$$

where we used Poincaré's inequality. Now, recall that

$$\frac{\delta F}{\delta u}[v]_x = J_x * v - \frac{v_x}{(1-v^2)} \qquad \text{so} \qquad \left|\frac{\delta F}{\delta u}[v]_x\right|_2 \leqslant C', \qquad \text{for} \quad t \in [0, \delta]$$

and,

$$\left[\sup_x \frac{1}{(1-v^2(x,t))}\right] \leqslant C \qquad \text{for} \quad t \in [0, \delta]$$

Thus

$$\frac{1}{2}\int_0^\delta \int (1-v^2)\, H_x^2 \, dx\, ds \leqslant \frac{2C}{\delta} |m - \tilde{u}|_2^2 + C'\delta \leqslant (2C + C')\,\delta$$

Now consider u which solves Eq. (6.10) between time 0 and $T(\delta)$ and which is defined as $u(x, T(\delta)+s) = v(x, s)$ for $s \in [0, \delta]$. Let now $\Phi(t) = u(T(\delta) + \delta - t)$ for $t \in [0, T(\delta) + \delta]$. For any $\varepsilon > 0$

$$I_{0,\, T(\delta)+\delta-\varepsilon}(\Phi) = I_{\varepsilon,\, T(\delta)}(u) + I_\delta(v)$$

Recall that u is smooth in $[\varepsilon, T(\delta)]$, so by using the definition of I_T

$$I_{\varepsilon,\, T(\delta)}(u) = F[u(\varepsilon)] - F[u(T(\delta))]$$

On the other hand,

$$I_{0,\varepsilon} = \frac{1}{2}\int_0^\varepsilon \int_{\mathscr{C}} \left[\frac{\delta F}{\delta u}[u]_x\right]^2 (1-u^2)\, dx$$

and by Fatou's Lemma

$$\begin{aligned}\int_0^\varepsilon \int_{\mathscr{C}} \left[\frac{\delta F}{\delta u}[u]_x\right]^2 (1-u^2)\, dx\, ds &\leqslant \liminf_{\delta \to 0} \int_\delta^\varepsilon \int_{\mathscr{C}} \left[\frac{\delta F}{\delta u}[u]_x\right]^2 (1-u^2)\, dx\, ds \\ &= \liminf_{\delta \to 0} F[u(\xi^*, \delta)] - F[u(\xi^*, \varepsilon)]\end{aligned}$$

Now, because ξ^* is in L^2, $\lim_{\varepsilon \to 0} |u(\xi^*, \varepsilon) - \xi^*|_2$, and by continuity of F in L^2, we have that $F[u(\xi^*, \varepsilon)] \to F[\xi^*]$ and

$$\varliminf_{\varepsilon \to 0} I_{0,\varepsilon}(u) = 0$$

Finally,

$$I_{0,\,T+\delta}(\Phi) \leqslant \tilde{C}\delta + F[\xi^*] - F[u(T(\delta))] \leqslant \tilde{C}\delta + F[\xi^*] - F[m]$$

The result follows as $\delta \to 0$. ∎

We need to establish now a lower bound for $\inf I_T(u)$ over paths such that $u(0) \in S_\gamma$, $u(0) \in ([m-\gamma, m+\gamma] + B_w)$ and $u(T) \in \partial D_\gamma + B_w$. If $H(T, \gamma, \delta)$ is defined as

$$\bigcup_{\xi \in \partial D_\gamma} \left\{ u \in D([0, T], S_\gamma) : \left| \left(u(0, \cdot) - \int u(0) \right) * \varphi_\delta \right|_\infty \right.$$
$$\left. \leqslant \delta,\ |(u(T, \cdot) - \xi) * \varphi_\delta|_\infty \leqslant \delta \right\}$$

then, we show that for any $\varepsilon > 0$ there is γ_o such that for $\gamma > \gamma_o$

$$\inf_{H(T,\gamma,\delta)} I_T(u) \geqslant (F[\xi^*] - F[m]) - \varepsilon \tag{6.11}$$

holds for large time T and δ small. Note that in any weak neighborhood of m, there will be a density ξ with $F[\xi] - F[m]$ large and thus $F[\xi^*] - F[\xi]$ much smaller than $F[\xi^*] - F[m]$. (for instance, when ξ is very oscillatory going from -1 to 1 with mean m.). However, such density will have a large value of $I_T(\xi)$, and they don't preclude (6.11) to hold.

Corollary 6.5. For any $\varepsilon > 0$ and $T > 0$, there is $\delta > 0$ such that (6.11) holds.

Proof. We first show, that there is a constant C independent of δ and T such that for any $u \in H(T, \gamma, \delta)$,

$$I_T(u) \geqslant I_T(\varphi_\delta * u) - C\sqrt{\delta T} \tag{6.12}$$

We will not need to look at the entire $H(T, \gamma, \delta)$. Indeed, if

$$K = \{u \in D([0, T], M_*) : I_T(u) \leqslant F[\xi^*] - F[m]\}$$

then in (6.11) we can replace $H(T, \gamma, \delta)$ by $K \cap H(T, \gamma, \delta)$. Recall that K is compact and there is $M < \infty$ such that for any $u \in K$,

$$\int_0^T \int (u_x)^2 \leqslant 1 + M$$

Also, recall that we can decompose $I_T(u) = \tilde{I}_T(u) + R_T(u)$, where $\tilde{I}_T(u)$ is convex. To show (6.12), we only need to see that

$$|R_T(u) - R_T(\varphi_\delta * u)| \leqslant C\sqrt{\delta T}$$

which by a simple exercise only require

$$\int_0^T \int |u(t,x) - \varphi_\delta * u(t,x)| \leqslant C\sqrt{\delta T}$$

Write,

$$\begin{aligned} |u(t,x) - \varphi_\delta * u(t,x)| &= \int \varphi(y) \int_0^{\delta y} |u_x(t, x-z)\, dz\, dy| \\ &\leqslant \int \varphi(y)\sqrt{|y|\,\delta}\, dy \left[\int_{\mathscr{C}} u_x^2(t,z)\, dz\right]^{1/2} \\ &\leqslant \sqrt{\delta}\left[\int_{\mathscr{C}} u_x^2(t,z)\, dz\right]^{1/2} \end{aligned}$$

After integrating over $[0, T]$, and applying the Cauchy–Schwarz inequality

$$\int_0^T \int |u(t,x) - \varphi_\delta * u(t,x)| \leqslant \sqrt{\delta T}\left[\int_0^T \int u_x^2(t,z)\, dz\right]^{1/2}$$

It is easy to conclude that for any $u \in K$,

$$I_T(u) \geqslant I_T(\varphi_\delta * u) - C\sqrt{\delta T}$$

Now, ε and T are fixed. There is δ such that

$$\forall \xi, \quad \xi' \in M_*: |\xi - \xi'|_\infty \leqslant \delta \Rightarrow |F[\xi] - F[\xi']| \leqslant \varepsilon \tag{6.13}$$

by uniform continuity of F in the supremum norm, and

$$F(\varphi_\delta * \xi^*) \geqslant F(\xi^*) - \varepsilon \tag{6.14}$$

by continuity of F in L^2. Now let $u \in H(T, \gamma, \delta) \cap K$, such that $I_T(u) - \varepsilon \leqslant I_T(v)$ for $v \in H(T, \gamma, \delta)$. We have

$$
\begin{aligned}
I_T(u) &\geqslant I_T(\varphi_\delta * u) - C\sqrt{\delta T} \\
&\geqslant F[\varphi_\delta * u(T)] - F[\varphi_\delta * u(0)] - C\sqrt{\delta T} \qquad [\text{by (6.9)}] \\
&\geqslant F[\varphi_\delta * \xi^*] - F\left[\int u(0)\right] - 2\varepsilon - C\sqrt{\delta T} \qquad [\text{by (6.13)}] \\
&\geqslant F[\xi^*] - F[m] - 3\varepsilon - C\sqrt{\delta T} \qquad (6.15)
\end{aligned}
$$

We can define, for any $\delta < \delta_o$, $T(u_o, \delta) = \inf\{t : u(u_o, t) \in U_\delta\}$. As long as $u_o \in D_\gamma$, we have $T(u_o, \delta) < \infty$. We show now that this time $T(u_o, \delta)$ is uniformly bounded.

Lemma 6.6. For $\delta < \delta_o$, there is T_o such that for any $u_o \in D_\gamma$ and $t \geqslant T_o$, $u(u_o, t) \in U_\delta$.

Proof. Let $a = \sup\{T(u_o, \delta) : u_o \in D_\gamma\}$ and $\{u_n\} \in D_\gamma$ such that $\lim T(u_n, \delta) = a$. There is $u_o \in D_\gamma$, such that $\{u_n\}$ converges to u_o weakly (possibly along a subsequence). Now, for any $t > T(u_o, \delta)$, $u(u_o, t)$ belongs to the L^2 interior of U_δ. By Lemma 3.2, for n large $u(u_n, t)$ belong to U_δ as well, so that $T(u_n, \delta) \leqslant t$ and $a < \infty$. ▌

Lemma 6.7. There are $a > 0$ and $T_o > 0$, such that if $u(t, \cdot) \in D_\gamma / U_\delta$ for all $t \in [0, T_o]$, then $I_{T_o}(u) > a$.

Proof. Let $T > 0$. Assume that $u(t, \cdot) \in D_\gamma / U_\delta$ for all $t \in [0, T]$ and that $I_T(u) < \infty$. We can define H by

$$
u_t = [\tfrac{1}{2}[u_x - \beta(1 - u^2) J_x * u] - H_x(1 - u^2)]_x
$$

and v satisfies (3.1) and $u(0, x) = v(0, x) = u_o(x) \in D_\gamma / U_\delta$. If $w = v - u$, then

$$
w_t = \tfrac{1}{2}[w_x - J_x * w(1 - v^2) - (J_x * u)\, w(u + v)]_x + [H_x(1 - u^2)]_x \qquad (6.16)
$$

and $w(0, x) = 0$. We multiple (6.16) by w and integrate by parts to obtain

$$
\begin{aligned}
&\frac{d}{dt} |w|_2^2 + |w_x|_2^2 \\
&\qquad = \int (J_x * w)\, w_x(1 - v^2) + \int (J_x * u)\, w w_x(u + v) - \int H_x(1 - u^2)\, w_x
\end{aligned}
$$

By applying the Cauchy–Schwarz inequality on terms of the right hand side,

$$\frac{d}{dt}|w|_2^2 \leqslant C\,|w|_2^2 + \int (H_x)^2\,(1-u^2)\,dx$$

Hence,

$$|w(T)|_2^2 \leqslant e^{CT} \int_0^T \int (H_x)^2\,(1-u^2) = e^{CT} I_T(u) \tag{6.17}$$

Now, by Lemma 6.6, there is T_o such that $v(T_o) \in U_{\delta/2}$, so

$$|w(T_o)|_2^2 \geqslant \delta/(2\gamma^2) \qquad \text{and} \qquad I_{T_o}(u) \geqslant e^{-CT_o}\,\delta/(2\gamma^2), \qquad [\text{by (6.17)}]$$

and we can set $a = \exp(-CT_o)\,\delta/2\gamma^2$. ∎

7. EXIT TIME

Let m and $\gamma > 0$ be such that $[m-\gamma, m+\gamma] \subset (\tilde{m}, m^+)$. If T is a continuity point of α, then for any $\varepsilon > 0$ let T_ε be another continuity point of α with $T_\varepsilon \in (T-\varepsilon, T)$. T_ε is needed because the Large Deviation estimates give informations only about small tubes around given paths. In establishing a lower bound for the exit time, we need to be sure that a path does not exit immediately, i.e., it must be at some distance (weakly) of the boundary of our domain. We recall that in Lemma 6.1 (ii) we showed that there is a neighborhood B_w such that $\Gamma_{T_\varepsilon,\gamma} + B_w \subset \Gamma_{T,\gamma}$. Now, we can define the stopping time for $u \in D([0,\infty), S_\gamma)$

$$\tau_\gamma(u) = \inf\{t : u(t) \notin \Gamma_{T,\gamma}\}$$

The main result of this section, is

Theorem 7.1. For $\varepsilon > 0$, there is γ_o such that if $\gamma < \gamma_o$, $T_\varepsilon < T - \varepsilon$ and N large enough, then for $\eta_N \in \{\mu_N(\eta_N) \in \Gamma_{T_\varepsilon,\gamma}\}$

$$\left|\frac{1}{N}\log E_N^{\eta_N}[\tau_\gamma] - \inf_{\partial_m \Gamma_{T,\gamma}} (F[\xi] - F[m])\right| \leqslant \varepsilon \tag{7.1}$$

Proof. The proof follows closely the arguments of Theorem 4.1 in Chapter 4 of ref. 8.

Step 1. We show the upper bound part of (7.1).

For any $T>0$, N, γ and η_N,

$$E_N^{\eta_N}[\tau_\gamma] = T\int_0^\infty P_N^{\eta_N}(\tau_\gamma > tT)\,dt \leqslant T\sum_{n=0}^\infty P_N^{\eta_N}(\tau_\gamma > nT) \tag{7.2}$$

By definition, if $\tau_\gamma(u)>t$, then $u(s)\in\Gamma_{T,\gamma}$ for $s\leqslant t$. Thus, by conditioning w.r.t. $\mathscr{F}_{(n-1)T}$ and using the Markov property

$$\begin{aligned}
P_N^{\eta_N}(\tau_\gamma > nT) &= P_N^{\eta_N}(\forall t\in[0,nT]:\mu_N(\eta_N)(t)\in\Gamma_{T,\gamma})\\
&= E_N^{\eta_N}[E_N^{\eta_N}[\chi(\tau_\gamma>(n-1)\,T)\\
&\quad\times\chi(\forall t\in[(n-1)\,T,nT]:\mu_N(\eta_N)(t)\in\Gamma_{T,\gamma})\,\|\,\mathscr{F}_{(n-1)T}]]\\
&= E_N^{\eta_N}[\chi(\tau_\gamma>(n-1)\,T)\,E_N^{\eta((n-1)T)}[\chi(\tau_\gamma>T)]]\\
&\leqslant (\sup_{\Gamma_{T,\gamma}} P_N^{\eta_N}(\tau_\gamma>T))^n = (1-\inf_{\Gamma_{T,\gamma}} P_N^{\eta_N}(\tau_\gamma\leqslant T))^n
\end{aligned} \tag{7.3}$$

where $\sup_{\Gamma_{T,\gamma}}$ means the supremum over $\{\eta:\mu_N(\eta_N)\in\Gamma_{T,\gamma}\}$. Now, (7.2) and (7.3) imply that

$$E_N^{\eta_N}[\tau_\gamma] \leqslant \frac{T}{\inf_{\Gamma_{T,\gamma}} P_N^{\eta_N}(\tau_\gamma\leqslant T)}$$

For $\varepsilon>0$, we choose γ_o so that (6.4) holds and B_δ such that (6.11) holds. By Theorem 5.7, there is T_1 such that any configuration η with $\mu_N(\eta_N)\in\Gamma_{T,\gamma}$,

$$\inf_{\Gamma_{T\gamma}} P_N^{\eta_N}(\mu_N(\eta_N)(T_1)\in B_\delta)\geqslant 1/2 \tag{7.4}$$

Let ζ_ε be the density which appears in Corollary 6.3: we recall that $\zeta_\varepsilon\notin\Gamma_{T\gamma}$ but that $F[\zeta_\varepsilon]$ is very close to the infimum of F on the boundary of $\Gamma_{T\gamma}$. Also, let Φ the path build in the proof of Lemma 6.4 with $\Phi(0)\equiv m$ and $\Phi(T_2)=\zeta_\varepsilon$: Φ is a path whose rate functional is very close to the minimum over all path joining m to the boundary. The Large Deviation lower bound Corollary 5.8 establishes that for N large enough

$$\begin{aligned}
\inf_{B_\delta}\frac{1}{N}\log P_N^{\eta_N}(\tau_\gamma\leqslant T_2) &\geqslant \inf_{B_\delta}\frac{1}{N}\log P_N^{\eta_N}(|(\mu_N(\eta_N)(T_2)-\Phi(T_2))*\varphi_\delta|_\infty<\delta)\\
&\geqslant -\inf_{\partial_m\Gamma_{T,\gamma}}(F[\xi]-F[m])-\varepsilon
\end{aligned}$$

Now, let $T = T_1 + T_2$. By the Markov property

$$\begin{aligned}
P_N^{\eta_N}(\tau_\gamma \leqslant T) &= P_N^{\eta_N}(\exists t \in [0, T] : \mu_N(\eta_N)(t) \in \Gamma_{T,\gamma}^c) \\
&\geqslant E_N^{\eta_N}[\chi(\mu_N(\eta_N)(T_1) \in B_\delta) \\
&\quad \times E_N^{\eta_N}[\chi(\exists t \in (T_1, T] : \mu_N(\eta_N)(t) \in \Gamma_{T,\gamma}^c) \mid \mathscr{F}_{T_1}]] \\
&\geqslant P_N^{\eta_N}(\mu_N(\eta_N)(\eta(T_1) \in B_\delta) \inf_{B_\delta} P_N^{\eta_N}(\tau_\gamma \leqslant T_2) \\
&\geqslant \tfrac{1}{2} \exp(N[-\inf_{\partial_m \Gamma_{T,\gamma}} (F[\xi] - F[m]) - \varepsilon])
\end{aligned}$$

This completes the first inequality.

Step 2. For the lower bound, we can restrict η_N to be such that $E_N^{\eta_N}[\tau_\gamma] < \infty$. Let $\delta > 0$ and define

$$B = \left\{ u \in S_\gamma : \left| \left(u - \int u \right) * \varphi_\delta \right|_\infty < \delta \right\}$$

and

$$C = \left\{ u \in S_\gamma : \left| \left(u - \int u \right) * \varphi_\delta \right|_\infty \in (2\delta, 3\delta) \right\}$$

By Lemma 6.1 (ii), the interior of $\Gamma_{T,\gamma}$ is not empty. Thus, for $\delta > 0$ small enough C and B are in $\Gamma_{T,\gamma}$.

We will now use repeatedly the Strong Markov Property (SMP): since we are dealing with a Poisson jump process in finite volume, it is straightforward to see that such a process has the SMP.

Now, we define stopping times $\{S_k\}$ and $\{T_k\}$ on $D([0, \infty), S_\gamma)$ by induction. First $T_o = 0$ and for $k \geqslant 0$

$$S_k(u) = \inf\{t \geqslant T_k : u(t) \in C\}$$

$$T_{k+1}(u) = \inf\{t \geqslant S_k : u(t) \in B \cup \Gamma_{T,\gamma}^c\}$$

and $\nu(u) = \inf\{k > 0 : u(T_k) \in \Gamma_{T,\gamma^c}\}$.

It follows from Theorem 5.7 that for N large enough

$$\inf_{\Gamma_{T_\delta,\gamma}} P_N^{\eta_N}(\tau_\gamma > S_1) \geqslant \tfrac{1}{2} \tag{7.5}$$

Therefore, by using the SMP and choosing η_N such that $\mu_N(\eta_N) \in \Gamma_{T_\varepsilon, \gamma^c}\}$

$$\begin{aligned} E_N^{\eta_N}[\tau_\gamma] &\geqslant E_N^{\eta_N}[\tau_\gamma \chi(\tau_\gamma > S_1)] = E_N^{\eta_N}[\chi(\tau_\gamma > S_1)\, E_N^{\eta_N}[\tau_\gamma \mid \mathscr{F}_{S_1}]] \\ &\geqslant P_N^{\eta_N}(\tau_\gamma > S_1) \inf_B E_N^{\eta_N}[\tau_\gamma] \geqslant \tfrac{1}{2} \inf_B E_N^{\eta_N}[\tau_\gamma] \end{aligned}$$

Now, a path $\{\mu_N(\eta_N)(t), t \geqslant 0\}$ being only right continuous, it could when started in B jump over C making $\nu = \infty$. However, we will see that during $[0, \tau_\gamma]$, for typical values of τ_γ, most of the paths will be "almost continuous" and on those paths $\nu < \infty$ when $\tau_\gamma < \infty$. We think of $\delta' > 0$ as a fixed number and for any $\alpha < \beta$ define

$$A_{[\alpha, \beta]} = \{\eta : \sup_{|t-s| \leqslant \delta',\, t, s \in [\alpha, \beta]} |(\mu_N(\eta_N)(t) - \mu_N(\eta_N)(s)) * \varphi_\delta|_\infty \leqslant \delta\}$$

We establish below in Lemma 7.2 that

$$\sup_C P_N^{\eta_N}(A^c_{[0, \tau_\gamma]}) \leqslant \exp(-2N\Omega) \tag{7.6}$$

with

$$\Omega = \inf_{\partial_m \Gamma_{T, \gamma}} (F[\xi] - F[m])$$

Now,

$$\tau_\gamma \geqslant \sum_{n=1}^{\infty} \chi\left[\nu \geqslant n, \bigcap_{k=1}^{n-1} A_{[S_{k-1}, S_k]} \cap A_{[S_{n-1}, \tau_\gamma]}\right] (T_n - T_{n-1})$$

Indeed, if $\tau_\gamma < \infty$ and the paths are in $A_{[0, \tau_\gamma]}$, they must always cross C when going from B to $\Gamma^c_{T, \gamma}$ and therefore $\nu < \infty$ and $\tau_\gamma = T_\nu$.

Taking now expectation of nonnegative quantities, we can exchange sum and integrals

$$\begin{aligned} E_N^{\eta_N} \tau_\gamma &\geqslant \sum_{n=1}^{\infty} E_N^{\eta_N}\left[\chi\left[\nu \geqslant n, \bigcap_{k=1}^{n-1} A_{[S_{k-1}, S_k]}, S_{n-1} < \tau_\gamma\right]\right. \\ &\qquad \left.\times E_N^{\eta_N}[A_{[S_{n-1}, \tau_\gamma]}(T_n - S_{n-1}) \mid \mathscr{F}_{S_{n-1}}]\right] \\ &\geqslant \sum_{n=1}^{\infty} E_N^{\eta_N}\left[\chi\left[\nu \geqslant n, \bigcap_{k=1}^{n-1} A_{[S_{k-1}, S_k]}\right] E^{\eta(S_{n-1})}[A_{[0, \tau_\gamma]} T_1]\right] \end{aligned} \tag{7.7}$$

where we have used the SMP. Now, any path in $A_{[0,\tau_y]}$ needs at least a time δ' to go from C to B

$$\inf_C E_N^{\eta_N}[A_{[0,\tau_y]} T_1] \geqslant \delta' \inf_C P_N^{\eta_N}(A_{[0,\tau_y]})$$

Thus, noting that by definition $\mu_N(\eta_N)(S_{n-1}) \in C$, (7.7) can be written

$$E_N^{\eta_N}[\tau_y] \geqslant \delta \inf_C P_N^{\eta_N}(A_{[0,\tau_y]}) \sum_{n=1}^{\infty} P_N^{\eta_N}\left(v \geqslant n \bigcap_{k=1}^{n-1} A_{[S_{k-1}, S_k]}\right) \tag{7.8}$$

Now, for each term of the sum in (7.8)

$$\begin{aligned} P_N^{\eta_N}\left(v \geqslant n, \bigcap_{k=1}^{n-1} A_{[S_{k-1}, S_k]}\right) &= P_N^{\eta_N}\left(\bigcap_{k=1}^{n-1} \{\mu_N(\eta_N)(T_k) \in B\} \cap A_{[S_{k-1}, S_k]}\right) \\ &\geqslant [\inf_C P_N^{\eta_N}(\mu_N(\eta_N)(T_1) \in B, A_{[S_0, S_1]})]^{n-1} \end{aligned}$$

by using the SMP again.

Finally,

$$E_N^{\eta_N}[\tau_y] \geqslant \frac{\delta \inf_C P_N^{\eta_N}(A_{[0,\tau_y]})}{1 - \inf_C P_N^{\eta_N}(\mu_N(\eta_N)(T_1) \in B, A_{[0,\tau_y]})} \tag{7.9}$$

Now, to reduce those estimates to finite time Large Deviations estimate, we introduce the time T, that we will choose appropriately latter and rewrite the denominator of (7.9) as

$$\begin{aligned} 1 &- \inf_C P_N^{\eta_N}(\mu_N(\eta_N)(T_1) \in B, A_{[0,\tau_y]}) \\ &\leqslant \sup_C (P_N^{\eta_N}(\mu_N(\eta_N)(T_1) \notin B, A_{[0,\tau_y]}) + P_N^{\eta_N}(A^c_{[0,\tau_y]})) \\ &\leqslant \sup_C P_N^{\eta_N}(T_1 = \tau_y < T, A_{[0,\tau_y]}) + \sup_C P_N^{\eta_N}(T_1 = \tau_y \geqslant T, A_{[0,\tau_y]}) \\ &\quad + \sup_C P_N^{\eta_N}(A^c_{[0,\tau_y]}) \end{aligned}$$

Also, we note that

$$\begin{aligned} &P_N^{\eta_N}(T_1 = \tau_y \geqslant T, A_{[0,\tau_y]}) \\ &\qquad \leqslant P_N^{\eta_N}(\mu_N(\eta_N)(t) \notin B, t \in [0, T]) \\ &P_N^{\eta_N}(T_1 = \tau_y < T, A_{[0,\tau_y]}) \\ &\qquad \leqslant P_N^{\eta_N}(\mu_N(\eta_N)(t) \in (\Gamma_{T,y} + \bar{B}) \cap (\text{int } \Gamma_{T,y})^c, \forall t \in [0, T]) \end{aligned}$$

To complete the argument it would suffice that there exists $T>0$, δ' and N_o such that $\forall N>N_o$

$$\sup_{C} P_N^{\eta_N}(\mu_N(\eta_N)(t) \notin B),\ t\in[0,T])$$

$$\leqslant \exp(-2N\Omega)$$

$$\sup_{C} P_N^{\eta_N}(\mu_N(\eta_N)(t) \in (\Gamma_{T,\gamma}+\bar{B}) \cap (\operatorname{int} \Gamma_{T,\gamma})^c,\ \forall t\in[0,T])$$

$$\leqslant \exp(-N[\Omega+\varepsilon])$$

This is established by using Theorem 5.11, Corollary 6.5 and Lemma 7.3 below. ▮

Lemma 7.2. For any $\delta>0$ and $a>0$, there is $\delta'>0$, and N_o such that

$$\forall N>N_o \sup_{\Gamma_{T,\gamma}} P_N^{\eta_N}(A^c_{[0,\tau_y]})) \leqslant \frac{1}{\delta'} e^{-aN}$$

Proof. First define

$$B(\delta,\delta',t)=\{\sup_{t<s<t+\delta'} |(\mu_N(\eta_N)(t)-\mu_N(\eta_N)(s)) * \varphi_\delta|_\infty > \delta\}$$

Now, we recall that from Step 1 of the proof Theorem 5.7 there is $\beta>0$

$$\sup_{\Gamma_{T,\gamma}} e_N^{\eta_N}[\tau_y] \leqslant e^{\beta N}$$

Now, for any $\alpha>0$

$$\begin{aligned}
P_N^{\eta_N}(A^c_{[0,\tau_y]}) &\leqslant P_N^{\eta_N}(A^c_{[0,\tau_y]} \cap \{\tau_y<e^{\alpha N}\}) + P_N^{\eta_N}(\tau_y \geqslant e^{\alpha N}) \\
&\leqslant P_N^{\eta_N}(A^c_{[0,e^{\alpha N}]}) + \frac{E_N^{\eta_N}[\tau_y]}{e^{\alpha N}} \qquad [\text{Chebychev}] \\
&\leqslant P_N^{\eta_N}\left(\bigcup_{n=0}^{K} [B(\varepsilon/2, 2\delta', 2\delta' n) \cup B(\varepsilon/2, 2\delta', 2\delta' n+\delta')]\right) \\
&\leqslant \frac{e^{\alpha N}}{2\delta'} \sup_{\eta} P_N^{\eta_N}(B(\varepsilon/2, 2\delta', 0)) + e^{(\beta-\alpha)N}
\end{aligned}$$

Now, similarly to ref. 14 Eq. (4.10), there is $c>0$ such that for any $l>0$, η_N and N large

$$P_N^{\eta_N}(B(\varepsilon, d)) \leqslant \exp(N(c\lambda^2\,\delta' - \varepsilon\lambda))$$

Hence, optimizing over l

$$P_N^{\eta_N}(A^c_{[0,\tau_\gamma]})) \leqslant e^{(\beta-\alpha)N} + \frac{1}{\delta'}\,e^{N(\alpha - \varepsilon^2/(4c\,\delta'))}$$

We can choose α and δ' such that $\beta-\alpha = -a$ and $(\alpha - \varepsilon^2/(4c\,\delta')) = -a$. This just requires that $\delta' \leqslant \varepsilon^2/(4c(2a+\beta))$. The result follows easily. ∎

Let B be an open neighborhood of $[m-\gamma, m+\gamma] \in S_\gamma$ and $D = \Gamma_{T,\gamma} + \bar{B}$. Note that D is closed. Now, we define E_1 and E_2 in $D([0, T], S_\gamma)$ by

$$E_1 = \{u \in D([0, T], S_\gamma) : u(0) \in \bar{B}, \exists t \leqslant T\; u(t) \in D \cap (\text{int}\, \Gamma_{T,\gamma})^c\}$$

$$E_2 = \{u \in D([0, T], S_\gamma) : u(t) \in B^c, \forall t \leqslant T\}$$

Lemma 7.3. For any $M>0$,

$$\{u : I_T(u) \leqslant M\} \cap \bar{E}_i \subset E_i, \qquad i=1,2 \tag{7.10}$$

and

$$-\frac{1}{N}\log(\sup_\eta P_N^{\eta_N}(\mu_N(\eta_N) \in E_i)) \geqslant \inf_{E_i} I_T \tag{7.11}$$

Proof. The arguments for E_1 and E_2 are similar so we show it only for E_1. Let $u_n \in E_1$ converging to u in the Skorohod topology and $I_T(u) \leqslant M$. We recall that this means that for $\varepsilon>0$ there is n_o such that for $n>n_o$

$$\inf_g \sup_{t\leqslant T} d(u(g(t)), u_n(t)) < \varepsilon \tag{7.12}$$

where g is an increasing bijection of $[0, T]$ and d is any metric for the weak topology.[7] As $I_T(u)<\infty$, u is continuous in time so it is enough to show that

$$\inf_{t\leqslant T}\; \inf_{D\cap(\text{int}\,\Gamma_{T,\gamma})^c} d(v, u(t)) = 0$$

But because $u_n \in E_1$, (7.12) means exactly that. Observe now that Theorem 5.11 gives that

$$-\frac{1}{N}\log(\sup_{\eta} P_N^{\eta_N}(\mu_N(\eta_N) \in E_i)) \geqslant \inf_{\bar{E}_i} I_T \geqslant \lim_{M \to \infty} \in f_{\bar{E}_i \cap \{I_T \leqslant M\}} I_T$$
$$\geqslant \inf_{E_i} I_T$$

where, in the last step, we used (7.10). ∎

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